SMALL BOUND ISOMORPHISMS OF THE DOMAIN OF A CLOSED ∗-DERIVATION

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Abstract. The domain \( \mathcal{D} \) of a closed ∗-derivation \( \delta \) in \( C(K) \) (\( K \): a compact Hausdorff space) is a generalization of the space \( C^{(1)}[0,1] \) of differentiable functions on \([0,1]\). In this paper, a problem proposed by Jarosz (1985) is studied in the context of derivations instead of \( C^{(1)}[0,1] \).

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Let \( K_1 \) and \( K_2 \) be two compact Hausdorff spaces. \( C(K_i) \) denotes a space of all complex valued continuous functions on \( K_i \) (\( i = 1,2 \)). Let \( T \) be a surjective linear isometry from \( C(K_1) \) to \( C(K_2) \). Then the Banach-Stone theorem states that there exist a homeomorphism \( \tau \) from \( K_2 \) to \( K_1 \) and a function \( w \) in \( C(K_2) \) with \( |w(y)| = 1 \) (\( y \in K_2 \)) such that
\[
T f(y) = w(y) f(\tau(y)) \quad \text{for} \quad f \in C(K_1), \ y \in K_2.
\]
That is, the existence of a surjective linear isometry between \( C(K_1) \) and \( C(K_2) \) implies that \( K_1 \) and \( K_2 \) are homeomorphic. Amir [1] and Cambern [2] extended this theorem from this viewpoint as follows.

Theorem 1 (see [1, 2]). If there is a surjective linear isomorphism \( T : C(K_1) \rightarrow C(K_2) \) such that \( \|T\| \cdot \|T^{-1}\| < 2 \), then \( K_1 \) and \( K_2 \) are homeomorphic.

Let \( X \) be a compact subset of the real line \( \mathbb{R} \) and \( C^{(1)}(X) \) be the space of continuously differentiable functions on \( X \) with the \( \Sigma \)-norm defined by \( \|f\|_{\Sigma} = \sup_{x \in X} |f(x)| + \sup_{x \in X} |f'(x)| \).

In [4], Jarosz proposed the following question: “Is there a positive \( \varepsilon \) such that for any compact subsets \( X,Y \) of the real line \( \mathbb{R} \) and any linear isomorphism \( T : C^{(1)}(X) \rightarrow C^{(1)}(Y) \), \( \|T\| \cdot \|T^{-1}\| < \varepsilon \) implies that \( X \) and \( Y \) are homeomorphic?”

In [5], Jun and Lee obtained some partial answers for this question.

Theorem 2 (see [5]). Let \( X \) and \( Y \) be compact subsets of \( \mathbb{R} \) and \( X \subset [a,b] \) and \( Y \subset [c,d] \). If \( T \) is a linear isomorphism between \( C^{(1)}(X) \) and \( C^{(1)}(Y) \) which satisfies
(i) \( f'(t) = 0 \), then \( (Tf)' = 0 \),
(ii) \( \|fg\| \leq \|TfTg\| \leq (1 + \varepsilon)^2 \|fg\| \),
(iii) \( \|f\| \leq \|Tf\| \leq (1 + \varepsilon) \|f\| \),
(iv) \( \varepsilon < \min\{1/49, 1/2(b-a+1), 1/2(c-d+1)\} \),
then \( X \) and \( Y \) are homeomorphic.
Theorem 3 [5]. Let $X$ and $Y$ be compact subsets of $\mathbb{R}$ and $X \subset \bigcup_{i=1}^{n}[a_i, b_i]$ $(a_i < b_i < a_{i+1})$ and $\max|b_i - a_i| < k$ and $Y \subset \bigcup_{j=1}^{m}[c_j, d_j]$ $(c_j < d_j < c_{j+1})$ and $\max|d_j - c_j| < k$. If $T$ is a linear map from $C^1(X)$ onto $C^1(Y)$ which satisfies

(i) $f'(t) = 0$ if and only if $(Tf)' = 0$,
(ii) $\|f\| \leq \|Tf\| \leq (1 + \varepsilon)\|f\|$,
(iii) $k < (4 - \sqrt{10})/6$ and $\varepsilon < 6k^2 - 8k + 1$,

then $X$ and $Y$ are homeomorphic.

In this paper, we consider this problem from another viewpoint. To the end, we recall a closed *-derivation.

Let $K$ be a compact Hausdorff space and $C(K)$ denotes the space of all complex-valued continuous functions on $K$ with the supremum norm $\| \cdot \|_\infty$. A closed *-derivation $\delta$ in $C(K)$ is a linear mapping in $C(K)$ satisfying the following conditions:

1. The domain $\mathcal{D}(\delta)$ of $\delta$ is a norm dense subalgebra of $C(K)$.
2. $\delta(fg) = \delta(f)g + f\delta(g)$ $(f, g \in \mathcal{D}(\delta))$.
3. If $f_n \in \mathcal{D}(\delta)$, $f_n \to f$, and $\delta(f_n) \to g$ implies $f \in \mathcal{D}(\delta)$ and $\delta(f) = g$ (i.e., $\delta$ is closed as a linear operator).
4. $f \in \mathcal{D}(\delta)$ implies $f^* \in \mathcal{D}(\delta)$ and $\delta(f^*) = \delta(f)^*$, where $f^*$ means the complex conjugate of $f$.

The differentiation $d/dt$ on the space $C^1([0, 1])$ of continuously differentiable functions on $[0, 1]$ is a typical example of closed *-derivations. For any closed *-derivation $\delta$ in $C(K)$, we may regard the domain $\mathcal{D}(\delta)$ of $\delta$ as a generalization of the Banach space $C^1([0, 1])$. Moreover, if $\mathcal{D}(\delta) = C(K)$, $\delta$ is bounded and hence $\delta = 0$.

Properties of the domains of closed *-derivations have been studied by many authors.

We summarize useful properties of closed *-derivations which is used later frequently without references.

Property 4 [7]. For $f = f^* \in \mathcal{D}(\delta)$ and $h \in C^1([-\|f\|_\infty, \|f\|_\infty])$, $h(f) = h \circ f \in \mathcal{D}(\delta)$ and $\delta(h(f)) = h'(f)\delta(f)$, where $h'$ means the derivative of $h$.

Property 5 [7]. If $f \in \mathcal{D}(\delta)$ is a constant in a neighborhood of $x \in K$, then $\delta(f)(x) = 0$.

Property 6 [7]. Let $J_1$ and $J_2$ be disjoint closed subsets of $K$. Then there is a function $f \in \mathcal{D}(\delta)$ such that

$$f = 0 \quad \text{on} \quad J_1, \quad f = 1 \quad \text{on} \quad J_2, \quad (0 \leq f \leq 1). \quad (2)$$

Now, for any fixed point $x \in K$, we define a linear functional $\eta_x \circ \delta$ on $\mathcal{D}(\delta)$ by

$$\eta_x \circ \delta(f) := \delta(f)(x) \quad (f \in \mathcal{D}(\delta)). \quad (3)$$

Let $K(\delta)$ be the set of $x \in K$ such that $\eta_x \circ \delta \neq 0$, i.e.,

$$K(\delta) = \{x \in K : \eta_x \circ \delta \neq 0\} = \{x \in K : \exists f \in \mathcal{D}(\delta) \text{ such that } \delta(f)(x) \neq 0\}. \quad (4)$$

Then $K(\delta)$ is an open subset of $K$. 
Throughout this paper, the norm $\| \cdot \|_{}$ in $S(\delta)$ is given by

$$\| f \|_{} := \| f \|_\infty + \| \delta(f) \|_\infty \quad (f \in S(\delta)).$$

Then we note that for $x_0 \in K(\delta)$, the norm of a linear functional $\eta_{x_0} \circ \delta$ is 1 (see [6]). In [6], we obtained the following result.

**Theorem 7.** Let $K_i$ be a compact Hausdorff space and let $\delta_i$ be a closed $*$-derivation in $C(K_i)$ ($i = 1, 2$). Let $T$ be a surjective linear isometry between $S(\delta_1)$ and $S(\delta_2)$. Then, there exist a homeomorphism $\tau$ from $K_2$ to $K_1$, $w_1 \in \ker(\delta_2)$ and a continuous function $w_2$ on $K_2(\delta_2)$ such that $\tau(K_2(\delta_2)) = K_1(\delta_1)$, $|w_1(y)| = 1$ for all $y \in K_2$, $|w_2(y)| = 1$ for all $y \in K_2(\delta_2)$,

$$\begin{align*}
(Tf)(y) &= w_1(y)f(\tau(y)) \quad \text{for } f \in S(\delta_1), \ y \in K_2, \\
\delta_2(Tf)(y) &= w_2(y)\delta_1(f)(\tau(y)) \quad \text{for } f \in S(\delta_1), \ y \in K_2(\delta_2).
\end{align*}$$

In this paper, we consider Jarosz’s problem in the same context as this theorem. We use the following notation, for a Banach space $B, B^*$ denotes the conjugate space of $B$. $B_1$ and $B_1^*$ denote the closed unit balls of $B$ and $B^*$, respectively. $T$ denotes the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in the complex plane.

We shall prove the following theorem.

**Theorem 8.** Let $K_1$ be a compact Hausdorff space satisfying the first countable axiom, and let $\delta_1$ be a closed $*$-derivation in $C(K_i)$ ($i = 1, 2$). If there exist a linear isomorphism $T$ of $S(\delta_1)$ onto $S(\delta_2)$ with $\|T\| \|T^{-1}\| < 2$ and $T, T^{-1}$ are bounded under the uniform norm, then $K_1(\delta_1)$ and $K_2(\delta_2)$ are homeomorphic. Moreover, if the range $\Re(\delta_i)$ contains $1$ ($i = 1, 2$), then $K_1$ and $K_2$ are homeomorphic.

The proof of this theorem is done along the line in [3].

Let $K$ be a compact Hausdorff space satisfying the first countable axiom and let $\delta$ be a closed $*$-derivation in $C(K)$.

The following two lemmas will be used in the rest of the paper.

**Lemma 9.** For $x_0 \in K(\delta)$, an open neighborhood $U$ of $x_0$ and $\varepsilon$ (0 < $\varepsilon$ < 1), there exists a function $f \in S(\delta)$ such that

$$\begin{align*}
\| f \| &\leq 1, \quad \| f \|_\infty \leq \varepsilon, \quad f(x_0) = 0, \\
f = \delta(f) = 0 &\quad \text{on } K \setminus U, \quad 1 > \| \delta(f)(x_0) \| > 1 - \varepsilon.
\end{align*}$$

**Proof.** We take an open neighborhood $V$ of $x_0$ such that $V \subset U$ and take a function $g \in S(\delta)$ such that

$$\begin{align*}
0 &\leq g \leq 1, \quad g(x_0) = 1, \quad g = 0 &\quad \text{on } K \setminus V.
\end{align*}$$

Then, $g = \delta(g) = 0$ on $K \setminus U$. Since $x_0 \in K(\delta)$, there is a function $g_\varepsilon (= g_\varepsilon^\ast) \in S(\delta)$ such that

$$\begin{align*}
\| g_\varepsilon \| &< 1, \quad 1 - \varepsilon = \| \eta_{x_0} \circ \delta \| - \varepsilon < \| \delta(g_\varepsilon)(x_0) \|.
\end{align*}$$
For \( c_\varepsilon := \min \left\{ (1 - \| \delta(g_\varepsilon) \|_\infty) / (1 + \| \delta(g) \|_\infty), \varepsilon \right\} \), there is a function \( h \in C^1([0, \| g_\varepsilon \|_\infty, \| g_\varepsilon \|_\infty]) \) such that

\[
\| h \|_\infty \leq c_\varepsilon, \quad h(g_\varepsilon(x_0)) = 0, \quad h'(g_\varepsilon(x_0)) = 1, \quad \| h' \|_\infty = 1. \tag{10}
\]

Then \( f := h(g_\varepsilon) \in \mathcal{D}(\delta) \) has all required properties in Lemma 9.

**Lemma 10.** For \( x_0 \in K(\delta) \) and \( \varepsilon \) \((0 < \varepsilon < 1)\), there exists a sequence \( \{ f_n \} \subset \mathcal{D}(\delta) \) such that

\[
\| f_n \| \leq 1, \quad \| f_n \|_\infty \leq \frac{1}{n}, \quad f_n(x_0) = 0, \quad \lim_{n \to \infty} \delta(f_n)(x) = 0 \quad (x \neq x_0), \quad 1 > |\delta(f_n)(x_0)| > 1 - \varepsilon,
\]

and \( d_{x_0} := \delta(f_n)(x_0) \) is independent of \( n \).

**Proof.** Since \( K \) satisfies the first countable axiom, there is a family \( \{ U_n \} \) of open neighborhood of \( x_0 \) such that \( U_{i+1} \subset U_i \) and \( \bigcap_{i=1}^\infty U_n = \{ x_0 \} \). Then there exists a family \( \{ V_n \} \) of open neighborhood of \( x_0 \) such that \( \bigcap_{i=1}^\infty V_n \subset U_n \), and there is \( g_n \in \mathcal{D}(\delta) \) such that

\[
g_n(x_0) = 1, \quad 0 \leq g_n \leq 1, \quad g_n = 0 \quad \text{on} \quad K \setminus V_n.
\tag{12}
\]

Then \( g_n = \delta(g_n) = 0 \) on \( K \setminus U_n \). Since \( x_0 \) is in \( K(\delta) \), there is a function \( g_\varepsilon := g_\varepsilon^* \in \mathcal{D}(\delta) \) such that

\[
\| g_\varepsilon \| < 1, \quad 1 - \varepsilon = \| h_{x_0} \circ \delta \| - \varepsilon < |\delta(g_\varepsilon)(x_0)|.
\tag{13}
\]

For each \( c_n := \min \left\{ (1 - \| \delta(g_\varepsilon) \|_\infty) / (1 + \| \delta(g_n) \|_\infty), 1/n \right\} \), there is a function \( h_n \in C^1([-\| g_\varepsilon \|_\infty, \| g_\varepsilon \|_\infty]) \) such that

\[
\| h_n \|_\infty \leq c_n, \quad h_n(g_\varepsilon(x_0)) = 0, \quad h_n'(g_\varepsilon(x_0)) = 1, \quad \| h_n' \|_\infty = 1.
\tag{14}
\]

Then every \( f_n := h_n(g_\varepsilon)g_n \in \mathcal{D}(\delta) \) has the properties required in Lemma 10.

Let \( W \) be the compact Hausdorff space \( W = K \times K \times T \) with the product topology. For \( f \in \mathcal{D}(\delta) \), we define \( \hat{f} \in C(W) \) by

\[
\hat{f}(x, x', z) := zf(x) + \delta(f)(x'),
\tag{15}
\]

for \((x, x', z) \in W\). Then we have \( \| \hat{f} \|_\infty = \| f \|_\infty \).

**Proof of Theorem 7.** Let \( W_i := K_i \times K_i \times T \) and \( S_i = \{ \hat{f} \in C(W_i); f \in \mathcal{D}(\delta) \} \) (\( i = 1, 2 \)).

Define a linear isomorphism \( \tilde{T} \) of \( S_1 \) onto \( S_2 \) by

\[
\tilde{T}(\hat{f}) := \hat{f}(\cdot, \cdot, z_0) \quad (\hat{f} \in S_1).
\tag{16}
\]

Then \( \tilde{T} \) is well defined since \( f \to \hat{f} \) is a linear isomorphism.

We may assume that \( \| T^{-1} \| = 1 \) and \( 1 < \| T \| < 2 \). Then we have \( \| \tilde{T}^{-1} \| = \| T^{-1} \| = 1 \) and \( \| \tilde{T} \| = \| T \| < 2 \). For \((y_0, y_0', z_0) \in W_2\), let \( \Phi \) be a norm-preserving extension of \( \tilde{T}^*L(y_0, y_0', z_0) \) to \( C(W_1) \), where \( L(y_0, y_0', z_0) \) denotes the linear functional defined by
Then for Lemma 10. Then for for
there exists a regular Borel measure $\mu^{y_0, y_0', z_0}$ on $W_1$ such that $\|\mu^{y_0, y_0', z_0}\| = \|\Phi\| = \|T^*L_{(y_0,y_0',z_0)}\| < 2$ and

$$\Phi(h) = \int_{W_1} h \, d\mu^{y_0, y_0', z_0} \quad (h \in C(W_1)).$$

(17)

Hence we have

$$z_0(T,f)(y_0) + \delta_2(T,f)(y_0') = \int_{W_1} \tilde{f}(x,x',z) \, d\mu^{y_0, y_0', z_0}$$

$$= \int_{W_1} (zf(x) + \delta_1(f)(x')) \, d\mu^{y_0, y_0', z_0}$$

(18)

for $f \in D(\delta_1)$. \hfill \Box

In the following, we identify $\Phi$ and $\mu^{y_0, y_0', z_0}$,

$\mu^{x_0,x_0',z_0}$, where $(x_0,x_0',z_0) \in W_1$, is also defined in a similar way. Then we have

$\|\mu^{x_0,x_0',z_0}\| \leq 1.$

The following lemma shows that for $x_0 \in K_1(\delta_1), \mu^{y,y',z}(K_1 \times \{x_0\} \times T)$, where $(y,y',z) \in W_2$ depends on $y'$ only, that is, $\mu^{y,y',z}(K_1 \times \{x_0\} \times T)$ is independent of $y$, $z$, and any choice of norm-preserving extension of $T^*L_{(y,y',z)}$.

**Lemma 11.** (1) For $x_0 \in K_1(\delta_1)$ and $\varepsilon \ (0 < \varepsilon < 1)$, let $\{f_n\} \subset D(\delta_1)$ be a sequence in Lemma 10. Then for $(y,y',z) \in W_2$,

$$\mu^{y,y',z}(K_1 \times \{x_0\} \times T) = \left(\frac{1}{d_{x_0}}\right) \lim_{n \to \infty} \tilde{T}(f_n)(y,y',z)$$

$$= \left(\frac{1}{d_{x_0}}\right) \lim_{n \to \infty} \delta_2(T(f_n))(y').$$

(19)

(2) For $y_0 \in K_2(\delta_2)$ and $\varepsilon \ (0 < \varepsilon < 1)$, let $\{g_n\} \subset D(\delta_2)$ be a sequence in Lemma 10. Then for $(x,x',z) \in W_1$,

$$\mu^{x,x',z}(K_2 \times \{y_0\} \times T) = \left(\frac{1}{d_{y_0}}\right) \lim_{n \to \infty} \tilde{T}^{-1}(g_n)(x,x',z)$$

$$= \left(\frac{1}{d_{y_0}}\right) \lim_{n \to \infty} \delta_1(T^{-1}(g_n))(x').$$

(20)

**Proof.** (1) Let $\mu^{y,y',z}$ be a norm-preserving extension of $T^*L_{(y,y',z)}$.

$$\lim_{n \to \infty} \tilde{T}(f_n)(y,y',z) = \lim_{n \to \infty} \int_{W_1} f_n \, d\mu^{y,y',z} = \int_{W_1} \lim_{n \to \infty} f_n \, d\mu^{y,y',z}$$

$$= \int_{K_1 \times \{x_0\} \times T} d_{x_0} \, d\mu^{y,y',z} = d_{x_0} \mu^{y,y',z}(K_1 \times \{x_0\} \times T).$$

(21)

From the uniform boundedness of $T$,

$$\lim_{n \to \infty} \tilde{T}(f_n)(y,y',z) = \lim_{n \to \infty} \left( z(Tf_n)(y') + \delta_2(Tf_n)(y') \right) = \lim_{n \to \infty} \delta_2(Tf_n)(y').$$

(22)
Thus, we have
\[
d_{x_0}\mu^{y,y',z}(K_1 \times \{x_0\} \times T) = \lim_{n \to \infty} \delta_2(Tf_n)(y')
\]
which implies that for \(x_0 \in K_1(\delta_1)\), \(\mu^{y,y',z}(K_1 \times \{x_0\} \times T)\) depends on \(y' \in K_2\) only.

The statement (2) is also shown by the same argument as above.

Now, let \(M_1\) be any real number with \((1 <) \|T\| < 2M_1 < 2\). Let \(\tilde{K}_2 := \{y \in K_2 : \exists x \in K_1 \text{ such that } |\mu^{y,y',z}(K_1 \times \{x\} \times T)| > M_1 \text{ for every } z \in T \text{ and every norm-preserving extension } \mu^{y,y',z} \text{ of } \tilde{T}^*L_{(y,y',z)}\}.\) Since \(\|\mu^{y,y,z}\| = \|\tilde{T}^*L_{(y,y',z)}\| \leq \|T\| < 2M_1\), for \(y \in \tilde{K}_2\), there can be at most one \(x \in K_1\) with the property in the definition of \(\tilde{K}_2\). Thus the map \(\rho_1\) of \(\tilde{K}_2\) to \(K_1\) is well defined by \(\rho_1(y) := x\) if \(x\) is related to \(y\) as above.

Next, we set \(M_2 := 1/(2M_1)\). Let \(\tilde{K}_1 := \{x \in K_1 : \exists y \in K_2 \text{ such that } |\mu^{x,x',z}(K_2 \times \{y\} \times T)| > M_2 \text{ for every } z \in T \text{ and for every norm-preserving extension } \mu^{x,x',z} \text{ of } (\tilde{T}^{-1})^*L_{(x,x',z)}\}.\) Since \(\|\mu^{x,x',z}\| = \|(\tilde{T}^{-1})^*L_{(x,x',z)}\| < \|T^{-1}\| \leq 1\), for \(x \in \tilde{K}_1\), there can be at most one \(y \in K_2\) with the property in the definition of \(\tilde{K}_1\). Thus, the map \(\rho_2\) of \(\tilde{K}_1\) to \(K_2\) is well defined by \(\rho_2(x) := y\) if \(y\) is related to \(x\) as above.

The following lemma shows that \(\tilde{K}_1\) contains sufficiently many elements (hence, is nonempty).

**Lemma 12.** (1) For \(x_0 \in K_1(\delta_1)\), there exists \(y_0 \in \tilde{K}_2 \cap K_2(\delta_2)\) such that \(\rho_1(y_0) = x_0\).
(2) For \(y_0 \in K_2(\delta_2)\), there exists \(x_0 \in \tilde{K}_1 \cap K_1(\delta_1)\) such that \(\rho_2(x_0) = y_0\).

**Proof.** (1) For \(x_0 \in K_1(\delta_1)\), there exists a family \(\{f_n\} \subset \mathcal{D}(\delta_1)\) in Lemma 10 such that
\[
\begin{align*}
|f_n| &\leq 1, \\
\|f_n\| &\leq \frac{1}{n}, \\
f_n(x_0) &\rightarrow 0,
\end{align*}
\]
where \(d_{x_0} = \delta_1(f_n)(x_0)\). If \(\lim_{n \to \infty} |\tilde{T}(\tilde{f}_n)(y,y',z)| \leq M_1\) for every \((y,y',z) \in W_2\), then
\[
1 - \varepsilon < |d_{x_0}| = \lim_{n \to \infty} |f_n(x_0) + \delta_1(f_n)(x_0)| = \lim_{n \to \infty} |\tilde{f}_n(x_0,x_0,1)|
\]
Then, from Lemma 11 we have for arbitrary $z \in T$ and any norm-preserving extension $\mu^{y_0,y_0'}_{T}$ of $\tilde{T}^*L(y_0,y_0',z)$,

$$M_1 < \lim_{n \to \infty} |\tilde{T}(\tilde{f}_n)(y_0,y_0',z_0)| = \lim_{n \to \infty} |\delta_2(T\tilde{f}_n)(y_0')| = \lim_{n \to \infty} |\tilde{T}(\tilde{f}_n)(y_0,y_0',z_0)| = |d_{x_0} \mu^{y_0,y_0'}_{T_0}(K_n \times \{x_0\} \times T)| \leq 1.$$  

(27)

Thus, $y_0' \in K_2 \cap K_2(\delta_2)$ and $\rho_1(y_0') = x_0$.

Now, we state another important lemma which holds without the first countability axiom.

**Lemma 13.** If $x_0 \in \tilde{K}_1$ and $\rho_2(x_0) \in K_2(\delta_2)$, then $x_0 \in K_1(\delta_1)$.

**Proof.** Let $\mu^{x_0,x_0,1}$ be a norm-preserving extension of $(\tilde{T}^{-1})^*L(x_0,x_0,1)$. Since $\mu^{x_0,x_0,1}$ is regular, since for all $\varepsilon$ such that $0 < \varepsilon < M_2/|T|$, there is an open neighborhood $U_\varepsilon$ of $\rho_2(x_0)$ such that

$$|\mu^{x_0,x_0,1}|(K_2 \times (U_\varepsilon \setminus \{\rho_2(x_0)\}) \times T) < \varepsilon.$$  

(28)

For $\varepsilon, U_\varepsilon$ and $\rho_2(x_0)$, we take a function $f \in \mathcal{Y}(\delta_2)$ in Lemma 9, then

$$\|f\| \leq 1, \quad \|f\|_\infty \leq \varepsilon, \quad f(\rho_2(x_0)) = 0,$$

$$f = \delta_2(f) = 0 \quad \text{on} \quad K_2 \setminus U_\varepsilon, \quad 1 > |\delta_2(f)(\rho_2(x_0))| > 1 - \varepsilon.$$  

(29)

Since

$$\left| \int_{K_2 \times (\rho_2(x_0)) \times T} zf(y) d\mu^{x_0,x_0,1} \right| \leq \|f\|_\infty \|\mu^{x_0,x_0,1}\| \leq \varepsilon,$$

$$\int_{K_2 \times (\rho_2(x_0)) \times T} \delta_2(f)(\rho_2(x_0)) d\mu^{x_0,x_0,1} = |\delta_2(f)(\rho_2(x_0))| \|\mu^{x_0,x_0,1}\| |(K_2 \times (\rho_2(x_0)) \times T)| > (1 - \varepsilon)M_2,$$

we have

$$\left| \int_{K_2 \times (\rho_2(x_0)) \times T} \tilde{f} d\mu^{x_0,x_0,1} \right| \geq \left| \int_{K_2 \times (\rho_2(x_0)) \times T} \delta_2(f)(\rho_2(x_0)) d\mu^{x_0,x_0,1} \right|$$

$$- \left| \int_{K_2 \times (\rho_2(x_0)) \times T} zf(y) d\mu^{x_0,x_0,1} \right| > (1 - \varepsilon)M_2 - \varepsilon > 0.$$  

(30)

From this and

$$\left| \int_{K_2 \times (U_\varepsilon \setminus \{\rho_2(x_0)\}) \times T} \tilde{f} d\mu^{x_0,x_0,1} \right| \leq \|\tilde{f}\|_\infty \|\mu^{x_0,x_0,1}\| |(K_2 \times (U_\varepsilon \setminus \{\rho_2(x_0)\}) \times T)| \leq \varepsilon,$$  

(31)

(32)
we have
\[\left| \int_{K_2 \times U_2 \times T} \tilde{f} \, d\mu_{x_0, x_0, 1} \right| \leq \left| \int_{K_2 \times \{\rho_2(x_0)\} \times T} \tilde{f} \, d\mu_{x_0, x_0, 1} \right| - \left| \int_{K_2 \times (U_2 \setminus \{\rho_2(x_0)\}) \times T} \tilde{f} \, d\mu_{x_0, x_0, 1} \right| \]
\[> (1 - \varepsilon)M_2 - 2\varepsilon > 0.\]

Since
\[\left| \int_{K_2 \times (K_2 \setminus U_2) \times T} \tilde{f} \, d\mu_{x_0, x_0, 1} \right| = \left| \int_{K_2 \times (K_2 \setminus U_2) \times T} z f(y) \, d\mu_{x_0, x_0, 1} \right| \]
\[\leq \|f\|_\infty \|\mu_{x_0, x_0, 1}\| \leq \varepsilon,
\]
we get
\[\left| (T^{-1}\tilde{f})(x_0, x_0, 1) \right| = \left| (T^{-1})^* L_{(x_0, x_0, 1)}(\tilde{f}) \right| = \left| \int_{K_2 \times U_2 \times T} \tilde{f} \, d\mu_{x_0, x_0, 1} \right| \]
\[\geq \left| \int_{K_2 \times U_2 \times T} \tilde{f} \, d\mu_{x_0, x_0, 1} \right| - \left| \int_{K_2 \times (K_2 \setminus U_2) \times T} \tilde{f} \, d\mu_{x_0, x_0, 1} \right| \]
\[\geq (1 - \varepsilon)M_2 - 3\varepsilon > 0.\]

Thus
\[|\delta_1(T^{-1}(f))(x_0)| = |T^{-1}(\tilde{f})(x_0, x_0, 1) - T^{-1}(f)(x_0)| \]
\[\geq |T^{-1}(\tilde{f})(x_0, x_0, 1)| - |T^{-1}(f)(x_0)| \]
\[\geq (1 - \varepsilon)M_2 - \varepsilon - \varepsilon\|T^{-1}\|_{\infty} > 0,
\]
that is, \(x_0 \in K_1(\delta_1)\). This completes the proof.

**Lemma 14.** If \(y_0 \in \hat{K}_2 \cap K_2(\delta_2)\), then \(\rho_1(y_0) \in \hat{K}_1 \cap K_1(\delta_1)\) and \(\rho_2(\rho_1(y_0)) = y_0\).

**Proof.** Let \(\rho_1(y_0) = x_0\) (\(y_0 \in \hat{K}_2 \cap K_2(\delta_2)\)). If \(x_0 \in \hat{K}_1\) and \(\rho_2(x_0) = y_0\), then \(x_0 \in K_1(\delta_1)\) from Lemma 13. Hence, suppose that either \(x_0\) is not in \(\hat{K}_1\) or \(x_0 \in \hat{K}_1\) and \(\rho_2(x_0) \neq y_0\). Then there exists \(z_0 \in T\) such that \(|\mu^{x_0, x_0, z_0}(K_2 \times \{y_0\} \times T)| \leq M_2\).

Let \(P := \sup\{|\mu^{x, x', z}(K_2 \times \{y_0\} \times T); (x, x', z) \in W_1| \leq 1\}\). Since \(y_0 \in K_2(\delta_2)\), we have \(P = \sup\{|\mu^{x, x', z}(K_2 \times \{y_0\} \times T); (x, x', z) \in W_1\}\) by Lemma 11. Since \(P > M_2\) by Lemma 12 and \(0 < \|T\| - M_1 < M_1\), there exists \((x_1, x_1, z_1) \in W_1\) such that
\[|\mu^{x_1, x_1, z_1}(K_2 \times \{y_0\} \times T)| > \max\{M_2, (\|T\| - M_1)P/M_1\}.\]
Then, for arbitrary \(z \in T\) and any norm-preserving extension \(\mu^{x_1, x_1, z}\),
\[|\mu^{x_1, x_1, z}(K_2 \times \{y_0\} \times T)| > M_2,
\]by Lemma 11. Thus, \(x_1 \in \hat{K}_1\), \(\rho_2(x_1) = y_0\), and \(x_1 \neq x_0\). Therefore, \(x_1 \in K_1(\delta_1)\) by Lemma 13. Since \(x_1 \neq x_0\), there exist \(y_1(\neq y_0) \in \hat{K}_2 \cap K_2(\delta_2)\) such that \(\rho_1(y_1) = x_1\).
by Lemma 12. For \( y_0 \in K_2(\delta_2) \) and \( \varepsilon (0 < \varepsilon < 1) \), there exists a family \( \{ g_n \} \subset \mathcal{D}(\delta_2) \) in Lemma 10. Then, since \( y_1 \neq y_0 \),

\[
0 = \lim_{n \to \infty} (z_1 g_n(y_1) + \delta_2(\varphi_n(y_1))) = \lim_{n \to \infty} \tilde{g}_n(y_1, y_1, z_1)
\]

\[
= \lim_{n \to \infty} \tilde{T}^* L_{y_1, y_1, z_1} (\tilde{T}^{-1}(\tilde{g}_n)) = \lim_{n \to \infty} \int_{\mathbb{W}_1} \tilde{T}^{-1}(\tilde{g}_n) d\mu^{y_1/y_1, z_1}
\]

\[
= \lim_{n \to \infty} \int_{K_1 \times \{ x_1 \} \times T} \tilde{T}^{-1}(\tilde{g}_n) d\mu^{y_1/y_1, z_1} + \lim_{n \to \infty} \int_{K_1 \times (K_1 \setminus \{ x_1 \}) \times T} \tilde{T}^{-1}(\tilde{g}_n) d\mu^{y_1/y_1, z_1}.
\]  

(39)

Now, by Lemma 11,

\[
\left\| \lim_{n \to \infty} \int_{K_1 \times \{ x_1 \} \times T} \tilde{T}^{-1}(\tilde{g}_n) d\mu^{y_1/y_1, z_1} \right\|
\]

\[
= \left\| \int_{K_1 \times \{ x_1 \} \times T} \lim_{n \to \infty} \tilde{T}^{-1}(\tilde{g}_n) d\mu^{y_1/y_1, z_1} \right\|
\]

\[
= \left\| \int_{K_1 \times \{ x_1 \} \times T} d_{y_0} \mu^{x/x, z} (K_2 \times \{ y_0 \} \times T) d\mu^{y_1/y_1, z_1} \right\|
\]

\[
= \left\| \int_{K_1 \times \{ x_1 \} \times T} d_{y_0} \mu^{x/x, z} (K_2 \times \{ y_0 \} \times T) d\mu^{y_1/y_1, z_1} \right\|
\]

\[
\geq \left\| d_{y_0} \right\| \| \| T \| - M_1 \| P \| T \| - M_1 \|.
\]  

(40)

On the other hand,

\[
\left\| \lim_{n \to \infty} \int_{K_1 \times (K_1 \setminus \{ x_1 \}) \times T} \tilde{T}^{-1}(\tilde{g}_n) d\mu^{y_1/y_1, z_1} \right\|
\]

\[
= \left\| \int_{K_1 \times (K_1 \setminus \{ x_1 \}) \times T} \lim_{n \to \infty} \tilde{T}^{-1}(\tilde{g}_n) d\mu^{y_1/y_1, z_1} \right\|
\]

\[
= \left\| \int_{K_1 \times (K_1 \setminus \{ x_1 \}) \times T} d_{y_0} \mu^{x/x, z} (K_2 \times \{ y_0 \} \times T) d\mu^{y_1/y_1, z_1} \right\|
\]

\[
\leq \left\| d_{y_0} \right\| \| \mu^{y_1/y_1, z_1} \| (K_1 \times (K_1 \setminus \{ x_1 \})) \times T
\]

\[
= \left\| d_{y_0} \right\| P (\mu^{y_1/y_1, z_1} \| (K_1 \times \{ x_1 \}) \times T) \times T)
\]

\[
\leq \left\| d_{y_0} \right\| P (\| T \| - \mu^{y_1/y_1, z_1} \| (K_1 \times \{ x_1 \}) \times T) < \left\| d_{y_0} \right\| P (\| T \| - M_1).
\]  

(41)

This contradicts to

\[
0 = \lim_{n \to \infty} \int_{K_1 \times \{ x_1 \} \times T} \tilde{T}^{-1}(\tilde{g}_n) d\mu^{y_1/y_1, z_1} + \lim_{n \to \infty} \int_{K_1 \times (K_1 \setminus \{ x_1 \}) \times T} \tilde{T}^{-1}(\tilde{g}_n) d\mu^{y_1/y_1, z_1}.
\]  

(42)

Thus \( x_0 \in \tilde{K}_1 \) and \( \gamma_0 = \rho_2(x_0) = \rho_2(\varphi_1(y_0)) \).

By Lemmas 12 and 14, we have \( K_1(\delta_1) \subseteq \rho_1(\tilde{K}_2 \cap K_2(\delta_2)) \subseteq \tilde{K}_1 \cap K_1(\delta_1) \subseteq K_1(\delta_1) \) and \( K_2(\delta_2) \subseteq \rho_2(\tilde{K}_1 \cap K_1(\delta_1)) = \rho_2(K_1(\delta_1)) = \rho_2(\rho_1(\tilde{K}_2 \cap K_2(\delta_2))) \subseteq \tilde{K}_2 \cap K_2(\delta_2) \subseteq K_2(\delta_2) \). Thus, \( K_1(\delta_1) \subseteq \tilde{K}_1, K_1(\delta_1) = \rho_1(\tilde{K}_2 \cap K_2(\delta_2)), \) and \( K_2(\delta_2) = \tilde{K}_2 \cap K_2(\delta_2) \subseteq \tilde{K}_2 \). Therefore, \( \rho_1(K_2(\delta_2)) = K_1(\delta_1) \) and \( \rho_2(K_1(\delta_1)) = K_2(\delta_2) \). Since \( \rho_2(\rho_1(y)) = y \) for \( y \in K_2(\delta_2) \) from Lemma 14, \( \rho_1 \) is injective on \( K_2(\delta_2) \). Moreover, we have \( \rho_1(\rho_2(x)) = x \) for \( x \in K_1(\delta_1) \) and hence \( \rho_2 \) is injective on \( K_1(\delta_1) \).
**Lemma 15.** $\rho_i$ is continuous on $K_i(\delta_i)$ ($i=1,2$).

**Proof.** We show that $\rho_1$ is continuous. Suppose that $\rho_1$ is discontinuous at $y_0 \in K_2(\delta_2)$. Then there exists a sequence $\{y_n\} \subset K_2(\delta_2)$ such that $y_n \to y_0 \in K_2(\delta_2)$, but $x_n := \rho_1(y_n)$ is not convergent to $\rho_1(y_0) = x_0$. There exists an open neighborhood $V_1(\subset K_1(\delta_1))$ of $x_0$ such that for every $n_0$ there is $n(\geq n_0)$ with $x_n$ outside $V_1$. Since $\mu^{y_0,y_0,1}$ is regular, for $\varepsilon > 0$ ($0 < \varepsilon < (2M_1 - \|T\|)/\varepsilon$) there exists an open neighborhood $U_1(\subset V_1)$ of $x_0$ such that

$$\left| \mu^{y_0,y_0,1} \left( K_1 \times (U_1 \setminus \{x_0\}) \times T \right) \right| < \varepsilon, \quad \overline{U}_1 \subset V_1. \quad (43)$$

For $x_0, U_1$, and $\varepsilon$, by Lemma 9, there exists a function $f \in \Theta(\delta_1)$ such that

$$\|f\| \leq 1, \quad \|f\|_{\infty} \leq \varepsilon, \quad f(x_0) = 0, \quad 1 > |\delta_1(f)(x_0)| > 1 - \varepsilon, \quad f = \delta_1(f) = 0 \quad \text{on} \ K_1 \setminus U_1. \quad (44)$$

Since

$$\left| \int_{K_1 \times \{x_0\} \times T} z f(x) \, d\mu^{y_0,y_0,1} \right| \leq \|f\|_{\infty} \|\mu^{y_0,y_0,1}\| \leq 2\varepsilon,$$

$$\left| \int_{K_1 \times \{x_0\} \times T} \delta_1(f)(x_0) \, d\mu^{y_0,y_0,1} \right| = |\delta_1(f)(x_0)| \|\mu^{y_0,y_0,1}\| \left( K_1 \times \{x_0\} \times T \right) > (1 - \varepsilon)M_1, \quad (45)$$

we have

$$\left| \int_{K_1 \times \{x_0\} \times T} \tilde{f} \, d\mu^{y_0,y_0,1} \right| > (1 - \varepsilon)M_1 - 2\varepsilon > \varepsilon. \quad (46)$$

From (46) and

$$\left| \int_{K_1 \times (U_1 \setminus \{x_0\}) \times T} \tilde{f} \, d\mu^{y_0,y_0,1} \right| \leq \|\tilde{f}\|_{\infty} \left| \mu^{y_0,y_0,1} \right| \left( K_1 \times (U_1 \setminus \{x_0\}) \times T \right) \leq \varepsilon, \quad (47)$$

we have

$$\left| \int_{K_1 \times U_1 \times T} \tilde{f} \, d\mu^{y_0,y_0,1} \right| \geq \left| \int_{K_1 \times \{x_0\} \times T} \tilde{f} \, d\mu^{y_0,y_0,1} \right| - \left| \int_{K_1 \times (U_1 \setminus \{x_0\}) \times T} \tilde{f} \, d\mu^{y_0,y_0,1} \right| \geq (1 - \varepsilon)M_1 - 3\varepsilon > 2\varepsilon \quad (48)$$

$$\left| \int_{K_1 \times (U_1 \setminus \{x_0\}) \times T} \tilde{f} \, d\mu^{y_0,y_0,1} \right| = \left| \int_{K_1 \times (U_1 \setminus \{x_0\}) \times T} z f(x) \, d\mu^{y_0,y_0,1} \right| \leq \|f\|_{\infty} \left| \mu^{y_0,y_0,1}\right| \leq 2\varepsilon.$$
This is a contradiction. Therefore, by the same way as above, we have

\[ |\tilde{T}(\tilde{f})(y_0, y_0, 1)| = |\tilde{T}^*L_{(y_0, y_0, 1)}(\tilde{f})| = \left| \int_{W_1} \tilde{f} d\mu^{y_0, y_0, 1} \right| \]

\[ \geq \left| \int_{K_1 \times U_1 \times T} \tilde{f} d\mu^{y_0, y_0, 1} \right| - \left| \int_{K_1 \times (K_1 \setminus U_1) \times T} \tilde{f} d\mu^{y_0, y_0, 1} \right| \]

\[ > (1 - \varepsilon)M_1 - 5\varepsilon > 0. \]

Now, since \( y_n - y_0 \) in \( K_2 \), then \((y_n, y_n, 1) \to (y_0, y_0, 1)\) in \( W_2 \). There exists \( n_0 \) such that \( \forall n > n_0 \) implies \( |\tilde{T}(\tilde{f})(y_n, y_n, 1)| > (1 - \varepsilon)M_1 - 5\varepsilon \). Fix \( n_1 \geq n_0 \) such that \( x_{n_1} = \rho_1(y_{n_1}) \) lies outside \( V_1 \). Since \( \mu^{y_n, y_{n_1}, 1} \) is regular, there exists an open neighborhood \( U_2(\subset K_1) \) of \( x_{n_1} \) such that

\[ |\mu^{y_{n_1}, y_{n_1}, 1}| = (K_1 \times (U_2 \setminus \{x_{n_1}\}) \times T) < \varepsilon, \quad U_1 \cap U_2 = \emptyset. \tag{50} \]

For \( x_{n_1}, U_2, \) and \( \varepsilon \), we take \( g(\in \mathcal{D}(\delta_1)) \) in Lemma 9 such that

\[ \|g\| \leq 1, \quad \|g\|_\infty \leq \varepsilon, \quad g(x_{n_1}) = 0, \]

\[ 1 > |\delta_1(g)(x_{n_1})| > 1 - \varepsilon, \quad g = \delta_1(g) = 0 \quad \text{on} \ K_1 \setminus U_2. \tag{51} \]

By the same way as above, we have

\[ \left| \int_{K_1 \times U_2 \times T} \tilde{g} d\mu^{y_{n_1}, y_{n_1}, 1} \right| > (1 - \varepsilon)M_1 - 3\varepsilon > 0, \]

\[ \left| \int_{K_1 \times (K_1 \setminus U_2) \times T} \tilde{g} d\mu^{y_{n_1}, y_{n_1}, 1} \right| \leq \|g\|_\infty \|\mu^{y_{n_1}, y_{n_1}, 1}\| \leq 2\varepsilon. \tag{52} \]

Then

\[ |\tilde{T}(\tilde{g})(y_{n_1}, y_{n_1}, 1)| = |\tilde{T}^*L_{(y_{n_1}, y_{n_1}, 1)}(\tilde{g})| = \left| \int_{W_1} \tilde{g} d\mu^{y_{n_1}, y_{n_1}, 1} \right| \]

\[ \geq \left| \int_{K_1 \times U_2 \times T} \tilde{g} d\mu^{y_{n_1}, y_{n_1}, 1} \right| - \left| \int_{K_1 \times (K_1 \setminus U_2) \times T} \tilde{g} d\mu^{y_{n_1}, y_{n_1}, 1} \right| \]

\[ > (1 - \varepsilon)M_1 - 5\varepsilon > 0. \tag{53} \]

Thus, if we choose a complex number \( \lambda_0 \in T \) such that \( \tilde{T}(\tilde{f})(y_{n_1}, y_{n_1}, 1) \) and \( \lambda_0 (\tilde{T}(\tilde{g}))(y_{n_1}, y_{n_1}, 1) \) have equal arguments, then

\[ \|f + \lambda_0 g\| = \max \{\|f\|_\infty, |\|g\|_\infty\|\} + \max \{\|\delta_1(f)\|_\infty, |\|\delta_1(g)\|_\infty\|\} \leq 1 + \varepsilon, \tag{54} \]

This is a contradiction. Therefore, \( \rho_1 \) is continuous on \( K_2(\delta_2) \). A similar argument shows that \( \rho_2 \) is continuous on \( K_1(\delta_2) \). \( \square \)

From Lemma 15, it follows that \( K_1(\delta_1) \) and \( K_2(\delta_2) \) are homeomorphic. Thus, all proofs of Theorem are completed.
There is not a nonzero closed $\ast$-derivation in $C(D)$ ($D$ is the Cantor set). However, we can obtain similar results for $C^{(1)}(X)$ ($X$ : a compact subset of $\mathbb{R}$) by the same way as above.

**References**


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