RECAPTURING SEMIGROUP COMPACTIFICATIONS OF A GROUP 
FROM THOSE OF ITS CLOSED NORMAL SUBGROUPS

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Abstract. We know that if \( S \) is a subsemigroup of a semitopological semigroup \( T \), and \( \mathcal{F} \) stands for one of the spaces \( \mathcal{A}, \mathcal{W}, \mathcal{M}, \mathcal{D}, \mathcal{E} \) or \( \mathcal{F} \mathcal{C} \), and \( (\epsilon, T^\mathcal{F}) \) denotes the canonical \( \mathcal{F} \)-compactification of \( T \), where \( T \) has the property that \( \mathcal{F}(S) = \mathcal{F}(T) |_S \), then \( (\epsilon |_S, \epsilon(S)) \) is an \( \mathcal{F} \)-compactification of \( S \). In this paper, we try to show the converse of this problem when \( T \) is a locally compact group and \( S \) is a closed normal subgroup of \( T \). In this way we construct various semigroup compactifications of \( T \) from the same type compactifications of \( S \).

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1. Introduction. For notation and terminology we follow Berglund et al. [2], as much as possible. Thus a topological semigroup is a semigroup \( S \) that is a Hausdorff topological space, the multiplication \( (s, t) \rightarrow st : S \times S \rightarrow S \) being continuous. \( S \) is called a semitopological semigroup if the multiplication is separately continuous, i.e., the maps \( \lambda_s : t \rightarrow st \) and \( \rho_s : t \rightarrow ts \) from \( S \) into \( S \) are continuous for each \( s \in S \). For \( S \) to be right topological only, the maps \( \rho_s \) are required to be continuous. Let \( G \) denote a locally compact group, and \( N \) is a closed normal subgroup of \( G \). A semigroup compactification of \( G \) is a pair \( (\varphi, X) \), where \( X \) is a compact right topological semigroup with identity 1, and \( \varphi : G \rightarrow X \) is a continuous homomorphism with \( \varphi(G) = X \), and \( \varphi(G) \subset \Lambda(X) = \{ x \in X | \lambda_x : X \rightarrow X \text{ is continuous} \} ; \Lambda(X) \) is called the topological center of \( X \). When there is no risk of confusion we often refer to \( (\varphi, X) \), or even to \( X \), as a compactification of \( G \).

A homomorphism from a compactification \( (\varphi, X) \) of \( S \) to a compactification \( (\varphi, Y) \) of \( S \) is a continuous function \( \theta : X \rightarrow Y \) such that \( \theta \circ \varphi = \varphi \). Then, \( Y \) is called a factor of \( X \), and \( X \) is an extension of \( Y \). A compactification with a given property \( P \) (such as that of being a semitopological semigroup or a topological group) is called a \( P \)-compactification. A universal \( P \)-compactification of \( S \) is a \( P \)-compactification which is an extension of every \( P \)-compactification of \( S \) (see [1, 2, 3]).

The \( C^* \)-algebra of all bounded continuous complex-valued functions on \( G \) is denoted by \( \mathcal{C}(G) \) with left and right translation operators, \( L_s \) and \( R_s \), defined for all \( s \in G \) by \( L_s f = f \circ \lambda_s \) and \( R_s f = f \circ \rho_s \). If \( \mathcal{A} \) is a \( C^* \)-subalgebra of \( \mathcal{C}(G) \) containing the constant functions, we denote by \( G^\mathcal{A} \) the spectrum of \( \mathcal{A} \) furnished with Gelfand topology (i.e., the weak* topology induced from \( \mathcal{A}^* \)); the natural map \( \epsilon : G \rightarrow G^\mathcal{A} \) is defined by \( \epsilon(s) f = f(s) \). When \( \mathcal{A} \) is left translation invariant (i.e., \( L_s f \in \mathcal{A} \) for all \( s \in G \) and \( f \in \mathcal{A} \)) we can define an action of \( G \) on \( G^\mathcal{A} \) by \( (s, \nu) \rightarrow \epsilon(s) \nu \), where \( (\epsilon(s) \nu)(f) = \nu(L_s f) \). Right
translation invariance and \( v \in \mathfrak{e}(s) \) are analogously defined (see [5, 7]).

A left translation invariant \( C^* \)-subalgebra of \( \mathfrak{e}(G) \) containing the constant functions is called left \( m \)-introverted if the function \( s \to (\nu f)(s) = v(L_s f) \) is in \( \mathfrak{d} \) for all \( f \in \mathfrak{d} \) and \( v \in \mathfrak{G} \); in this situation the product of \( \mu, \nu \in \mathfrak{G} \) can be defined by \( (\mu \nu)(f) = \mu(\nu f) \). This makes \( (e, \mathfrak{G}) \) a semigroup compactification of \( G \). The spaces of almost periodic, weakly almost periodic, left continuous and distal functions, which are denoted by \( \mathfrak{d} \mathfrak{A}, \mathfrak{WdA}, \mathfrak{L}, \mathfrak{A}, \) respectively, are left \( m \)-introverted. We refer the reader to [2, 5] for the one-to-one correspondence between compactifications of \( G \) and left \( m \)-introverted \( C^* \)-subalgebras of \( \mathfrak{e}(G) \), and also for a discussion of properties \( P \) of compactifications and associated universal mapping properties.

2. Main results. Let \( G \) be a locally compact group with a closed normal subgroup \( N \), and let \( (\varphi, X) \) be a compactification of \( N \). Let \( \sim \) be the equivalence relation on \( G \times X \) with equivalence classes \( \{(sr^{-1}, \varphi(r)x) \mid r \in N\} \). Thus

\[
(s, x) \sim (t, y) \text{ if and only if } t^{-1}s \in N \text{ and } \varphi(t^{-1}s)x = y.
\]

\( \pi : G \times X \to (G \times X)/\sim \) will denote the quotient map. Clearly \( \pi \) is one-to-one on \( \{e\} \times X \), so we can identify \( X \equiv \{e\} \times X \) with \( \pi(\{e\} \times X) \). It is important that \( (G \times X)/\sim \) is locally compact and Hausdorff. In this connection we have the following lemmas, which are stated in [6].

**Lemma 2.1.**

(i) The graph of \( \sim \) is closed.

(ii) \( \pi : (G \times X) \to (G \times X)/\sim \) is an open mapping.

(iii) Let \( K \) be a compact subset of \( G \) and let \( L = KN \), then \( \pi(K \times X) = \pi(L \times X) \).

This lemma has the following easy consequences.

**Lemma 2.2.** The quotient space \( (G \times X)/\sim \) is locally compact and Hausdorff.

**Lemma 2.3.** If \( G = KN \) for some compact subset \( K \) of \( G \), then \( (G \times X)/\sim \) is compact.

Let \( \mu : G \to (G \times X) \) be defined by \( \mu(s) = (s, 1) \), where 1 is the identity of \( X \). Then, \( \pi \circ \mu : G \to (G \times X)/\sim \) is continuous as a composition of two continuous functions, and \( \pi \circ \mu(G) = \pi(G \times \varphi(N)) \), since for each \( (s, \varphi(r)) \in G \times \varphi(N), (s, \varphi(r)) \sim (sr, 1) \), and \( \pi \circ \mu(sr) = \pi(sr, 1) = \pi(s, \varphi(r)) \). Furthermore, if \( \varphi \) is a homeomorphism of \( N \) into \( X \), then \( \pi \circ \mu \) is also a homeomorphism.

We now define \( \sigma_s(r) = s^{-1}rs \) for \( s \in G \) and \( r \in N \), it is obvious that \( \sigma_s : N \to N \) is a surjective homomorphism for each \( s \in G \).

**Definition 2.4.** A \( \mathcal{P} \)-compactification \( (\varphi, X) \) of \( N \) is said to be a conjugation invariant \( \mathcal{P} \)-compactification of \( N \) if \( (\varphi \circ \sigma_s, X) \) is a \( \mathcal{P} \)-compactification of \( N \) for each \( s \in G \). When we write \( \mathcal{P} \)-compactification instead of \( P \)-compactification, this means that we want to emphasize its conjugation invariance, see Corollary 2.7.

**Remark.** The reader may have noticed that, the definition of \( \mathcal{P} \)-conjugation invariant compactification is different from the compatibility of a compactification in [6], because if \( \mathcal{P} \) is a property of compactifications that is not invariant under homomorphism and \( (\psi, X) \) is a \( \mathcal{P} \)-compactification of \( N \) compatible with \( G \), then \( (\psi \circ \sigma_s, X) \) is a
compactification of $N$ which may not be a $\mathcal{P}$-compactification of $N$, thus $(\psi, X)$ can fail to be a $\mathcal{P}$-conjugation invariant compactification of $N$. On the other hand, if $(\psi, X)$ is a $\mathcal{P}$-conjugation invariant compactification of $N$, i.e., $(\psi \circ \sigma_s, X)$ is a $\mathcal{P}$-compactification of $N$ for each $s \in G$, it is not always true that $\sigma_s$ has an extension from $X$ to $X$.

**Lemma 2.5.** Let $G$ be a locally compact group, $N$ a closed normal subgroup, and $(\varphi, X)$ a conjugation invariant universal $\mathcal{P}$-compactification of $N$, then each $\sigma_s$ can be extended continuously to a mapping from $X$ to $X$.

**Proof.** By conjugation invariance of $(\varphi, X)$, $(\varphi \circ \sigma_s, X)$ is a $\mathcal{P}$-compactification of $N$, and by universality of $(\varphi, X)$ there exists a continuous homomorphism $\nu : X \to X$ such that $\varphi \circ \sigma_s = \nu \circ \varphi$ for each $s \in N$. This $\nu$ is the continuous function extending $\sigma_s$.

It is obvious that if $(\varphi, X)$ is a conjugation invariant universal $\mathcal{P}$-compactification of $N$, then each $\sigma_s$ determines a continuous transformation of $X$, for which we use the same notation $\sigma_s$.

**Corollary 2.6.** Let $N$ be contained in the center of $G$, then each compactification $(\varphi, X)$ of $N$ is conjugation invariant.

**Corollary 2.7.** Let $(\epsilon, N^\varphi)$ denote a universal $\mathcal{P}$-compactification of $N$ and let $\mathcal{P}$ be a purely algebraic property, then $(\epsilon, N^\varphi)$ is a conjugation invariant $\mathcal{P}$-compactification of $N$.

Notice our deviation from the usual notation.

**Corollary 2.8.** Let $(\varphi, X)$ be an $\mathcal{F}$-compactification of $N$, where $\mathcal{F}$ stands for either of the spaces $\mathcal{AP}$ and $\mathcal{WAP}$, then $(\varphi, X)$ is a conjugation invariant $\mathcal{F}$-compactification of $N$.

**Lemma 2.9.** Let $(\varphi, X)$ be a conjugation invariant $\mathcal{P}$-compactification of $N$, then for each $s \in G$, $\sigma_s$ is a continuous automorphism of $X$.

**Proof.** $\sigma_s$ is a homeomorphism of $X$ onto $X$ (since $\sigma_s(N) = N$ and $\sigma_s \sigma_s^{-1} = I$, the identity mapping). Now, we show that $\sigma_s$ is a homomorphism. Obviously,

$$\sigma_s(xy) = \sigma_s(x) \sigma_s(y) \quad \text{for each} \ x, y \in \varphi(N). \quad (2.2)$$

Since $X$ is a right topological semigroup with $\varphi(N) \subset \Lambda(X)$, we conclude that (2.2) holds for each $x \in \varphi(N), y \in X$. Then it follows that (2.2) holds for all $x, y \in X$, as required.

If $N$ is a closed subgroup of $G$, and $X$ is a conjugation invariant $\mathcal{P}$-compactification of $N$, then we can define a semidirect product structure on $G \times X$ by $(s, x)(t, y) = (st, \sigma_t(x)y)$, where $\sigma_t$ is the conjugation map.

**Lemma 2.10.** Let $G$ be a locally compact group with a closed normal subgroup $N$, and let $(\varphi, X)$ be a conjugation invariant $\mathcal{P}$-compactification of $N$, then $G \times X$ is a right topological semigroup. Furthermore, the map

$$(s, r) \mapsto (st, \varphi(\sigma_t(r)y)) : (G \times N) \times (G \times X) \to G \times X \quad (2.3)$$
is continuous, and the equivalence relation $\sim$ is a congruence on $G \times X$.

**Proof.** The continuity is an easy conclusion of Ellis theorem. Now, we show that $\sim$ is a congruence. Suppose $(s,x) \sim (t,y)$ and $(u,z) \in G \times X$, then $t^{-1}s \in N$ and $\phi(t^{-1}s)x = y$, so $(s,x)(u,z) = (su,\sigma_u(x)z)$ and $(t,y)(u,z) = (tu,\sigma_u(y)z)$.

On the other hand, $(su,\sigma_u(x)z) \sim (tu,\sigma_u(y)z)$ since $(tu)^{-1}su = u^{-1}t^{-1}su \in N$ and 

$$\phi((tu)^{-1}su)\sigma_u(x)z = \sigma_u(y)z,$$

thus

$$(s,x)(u,z) \sim (t,y)(u,z).$$

Similarly

$$(u,z)(s,x) \sim (u,z)(t,y).$$

The following theorem is an easy consequence of the previous corollaries and lemmas.

**Theorem 2.11.** Let $G$ be a locally compact group with a closed normal subgroup $N$, and let $(\phi,X)$ be a conjugation invariant compactification of $N$. Then $(G \times X)/\sim$ is a locally compact right topological semigroup, and a compactification of $G$, provided that $G = KN$ for some compact subset $K$ of $G$.

**Theorem 2.12.** The compactification $(\pi \circ \mu,(G \times X)/\sim)$ of $G$ described in the previous theorem has the following universal property; let $(\varphi,Y)$ be a semigroup compactification of $G$ such that $\varphi|N$ extends to a continuous homomorphism $\phi : X \to Y$ in such a way that for each $s \in G$ and $x \in X$,

$$\phi(\sigma_s(x)) = \varphi(s^{-1})\phi(x)\varphi(s),$$

then there exists a (unique) continuous homomorphism $\theta : (G \times X)/\sim \to Y$ such that $\theta \circ \pi \circ \mu = \phi$.

**Proof.** We define $\theta_0 : G \times N \to Y$ by $\theta_0(s,x) = \varphi(s)\phi(x)$, then $\theta_0$ is a continuous homomorphism which is constant on $\sim$-classes of $G \times X$. Now we take $\theta = \theta_0 \circ \pi$. 

**Theorem 2.13.** Let $N$ be a closed normal subgroup of $G$ with $G = KN$ for some compact subset $K$ of $G$. Suppose that $\mathfrak{P}$ is a property of compactifications such that $(\varphi|N,\overline{\varphi(N)})$ is a $\mathfrak{P}$-compactification of $N$ whenever $(\varphi,\overline{\varphi(G)})$ is a $\mathfrak{P}$-compactification of $G$. Suppose that $(\epsilon,N^\#)$ is a conjugation invariant universal $\mathfrak{P}$-compactification of $N$. If $(G \times N^\#)/\sim$ has the property $\mathfrak{P}$, then $(G \times N^\#)/\sim$ is the universal $\mathfrak{P}$-compactification of $G$.

**Proof.** We show that $(G \times N^\#)/\sim$ is the universal $\mathfrak{P}$-compactification of $G$. Let $(\varphi,X)$ be a $\mathfrak{P}$-compactification of $G$ such that $(\varphi|N,\overline{\varphi(N)})$ is a $\mathfrak{P}$-compactification of $N$, by the universal property of $N^\#$ there exists a continuous homomorphism $\phi : N^\# \to X$ such that $\phi \circ \epsilon = \varphi|N$, and we have $\phi(\sigma_s(x)) = \varphi(s^{-1})\phi(x)\varphi(s)$ for all $s \in G$ and $x \in N^\#$. Notice that we use two different scripts of the same letter to emphasize
their connection. Indeed, for fixed \( s \in G \), both sides represent homomorphisms of \( N^p \) into \( X \), both sides are continuous in \( x \), and coincide on the dense subspace \( N \). Now the map \( \varphi \times \phi : (G \times N^p) \to X \) defined by \((\varphi \times \phi)(s,x) = \varphi(s)\phi(x)\) is continuous and a homomorphism, since

\[
(\varphi \times \phi)((s,x)(t,y)) = (\varphi \times \phi)(st, \sigma_t(x)y) = \varphi(st)\phi(\sigma_t(x)y) = \varphi(s)\varphi(t)\phi(\sigma_t(x))\phi(y) = \varphi(s)\phi(x)\varphi(t)\phi(y) = \varphi \times \phi(s,x)\varphi \times \phi(t,y).
\]

(2.8)

Also \( \varphi \times \phi \) is constant on \( \sim \)-classes, thus the quotient of \( \varphi \times \phi \) gives a continuous homomorphism from \( (G \times N^p)/\sim \) to \( X \).

**Corollary 2.14.** Let \( N \) be a closed normal subgroup of \( G \) with \( G = KN \) for some compact subset \( K \) of \( G \), then

(i) \( (G \times N^p)/\sim \) is the universal \( \mathcal{L}^\mathcal{L} \)-compactification of \( G \).

(ii) \( (G \times N^3)/\sim \) is the universal \( \mathcal{D} \)-compactification of \( G \).

**Proof.** (i) Since \( (G \times N^p)/\sim \) is a compactification of \( G \), by Theorem 2.13, \( (G \times N^p)/\sim \) is the universal \( \mathcal{L}^\mathcal{L} \)-compactification of \( G \).

(ii) Since \( N^3 \) is a group, \( (G \times N^3)/\sim \), the quotient by a congruence of a semidirect product of groups is also a group, thus by Theorem 2.13 \( (G \times N^3)/\sim \) is the universal \( \mathcal{D} \)-compactification of \( G \).

In some situations, we want to be able to conclude that the right topological semigroup \( (G \times X)/\sim \) of Theorem 2.13 is also left topological. The following lemma can be helpful in this connection.

**Lemma 2.15.** Let \( G \) be a locally compact group with a closed normal subgroup \( N \) and let \( X \) be a universal conjugation invariant compactification of \( N \). Suppose that \( G = KN \) for some compact subset \( K \) of \( G \) and \( s \rightarrow \sigma_s(x) : G \times X \to X \) is continuous for all \( x \in N \). Then \( (G \times X)/\sim \) is semitopological.

**Proof.** Since \( (s,x) \rightarrow \sigma_s(x) : G \times X \to X \) is a group action, it is continuous by Ellis theorem, thus \( G \times X \) is semitopological semigroup and also \( (G \times X)/\sim = \pi(G \times X) \).

**Corollary 2.16.** Let \( G \) be a locally compact group with a closed normal subgroup \( N \), \( G = KN \) for some compact subset \( K \) of \( G \) and suppose that \( s \rightarrow \sigma_s(x) : G \to N^{W_{\mathfrak{a}^p}} \) is continuous for all \( x \in N^{W_{\mathfrak{a}^p}} \), then \( (G \times N^{W_{\mathfrak{a}^p}})/\sim \) is the universal semitopological semigroup compactification of \( G \).

**Proof.** Since \( N^{W_{\mathfrak{a}^p}} \) is a semitopological semigroup, by Lemma 2.15, \( (G \times N^{W_{\mathfrak{a}^p}})/\sim \) is semitopological semigroup. Thus by Theorem 2.13, \( (G \times N^{W_{\mathfrak{a}^p}})/\sim \) is the universal semitopological semigroup compactification of \( G \).

A similar argument yields the following corollary.

**Corollary 2.17.** Let \( G \) be a locally compact group with a closed normal subgroup \( N \), \( G = KN \) for some compact subset \( K \) of \( G \) and suppose that \( s \rightarrow \sigma_s(x) : G \to N^{\mathfrak{a}^p} \) is
continuous for all \( x \in N^{sp} \), then \((G \times N^{sp})/\sim\) is the universal topological semigroup compactification of \( G \).

**Lemma 2.18.** Let \( N \) be a closed normal subgroup of \( G \) with \( G = KN \) for some compact subset \( K \) of \( G \). Let \( \mathcal{F} \) and \( \mathcal{G} \) be left \( m \)-introverted subalgebras of \( \mathcal{C}(N) \) and \( \mathcal{C}(G) \), respectively. Then \( N^{sp} \) is a conjugation invariant \( \mathcal{F} \)-compactification of \( N \) if and only if 
\[ \mathcal{G}|_{N} = \mathcal{F} \text{ and } (G \times N^{sp})/\sim \text{ is the } \mathcal{G} \text{-compactification of } G. \]

**Proof.** Let \( \mathcal{G}|_{N} = \mathcal{F} \), we define \( \sigma_{s}(x)(f) \) for \( s \in G, x \in N^{sp} \) and \( f \in \mathcal{F} \) by \( \sigma_{s}(x)(f) = x(g \circ \sigma_{s}|_{N}) \), where \( g \in \mathcal{G}, g|_{N} = f \). Since every such extension \( g \) yields a \( g \circ \sigma_{s} \) agreeing with \( f \circ \sigma_{s} \) on \( N \), \( \sigma_{s}(x) \) is well defined. So \( N^{sp} \) is a conjugation invariant \( \mathcal{F} \)-compactification of \( N \).

Conversely, since the quotient map \( \pi: G \times N^{sp} \to (G \times N^{sp})/\sim \) is injective on the compact set \( N^{sp} \cong \{e\} \times N^{sp} \), it gives a topological isomorphism of \( N^{sp} \) into \( (G \times N^{sp})/\sim \cong G^{sp} \).

**Corollary 2.19.** Let \( G \) be a compact group with a closed normal subgroup \( N \), then 
(i) \((G \times N^{sp})/\sim \cong G^{sp} \).
(ii) \((G \times N^{sp})/\sim \cong G^{sp} \).

**Corollary 2.20.** Let \( N \) be a closed normal subgroup of a locally compact group \( G \) contained in the center of \( G \), then 
\[ (G \times N^{sp})/\sim \cong G^{sp}. \] (2.9)

The next example shows that the continuity of \( s \to \sigma_{s}(x) \) in Corollary 2.14 and Lemma 2.15 is an essential condition.

**Example 2.21.** Let \( G = \mathbb{C} \times \mathbb{T} \) be the Euclidean group of the plane with \((z, w)(z_{1}, w_{1}) = (z + wz_{1}, ww_{1})\) and \( N = \mathbb{C} \times \{1\} \), then \( N \) is a closed normal subgroup of \( G \) and \( \mathcal{A}(G)|_{N} \) is a proper subset of \( \mathcal{A}(N) \) \([4, 8]\), so by Lemma 2.15 \((G \times \mathbb{C}^{sp})/\sim \) is not the universal \( \mathcal{A} \)-compactification of \( G \), \( \mathbb{C}^{sp} \) is a conjugation invariant compactification of \( N \), so the continuity of \( s \to \sigma_{s} \) must fail to hold Lemma 2.15. From \([4, 8]\), we can similarly conclude that \((G \times \mathbb{C}^{sp})/\sim \) is not the universal \( W \mathcal{A} \)-compactification of \( G \) and that the continuity of \( s \to \sigma_{s} \), as required by Corollary 2.14, also fails to hold.

**References**


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