SOME INEQUALITIES IN $B(H)$

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ABSTRACT. Let $H$ denote a separable Hilbert space and let $B(H)$ be the space of bounded and linear operators from $H$ to $H$. We define a subspace $\Delta(A,B)$ of $B(H)$, and prove two inequalities between the distance to $\Delta(A,B)$ of each operator $T$ in $B(H)$, and the value $\sup\{\|A^nTB^n - T\| : n = 1, 2, \ldots\}$.

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1. Notations. Throughout this paper $H$ denotes a separable Hilbert space and $\{e_n\}_{n=1}^\infty$ an orthonormal basis. Let $L_A$ and $R_B$ be left and right translation operators on $B(H)$ for $A, B \in B(H)$, satisfying $\|A\| \leq 1$ and $\|B\| \leq 1$. Then the set $\Delta(A,B)$ is defined by

$$\Delta(A,B) = \{T \in B(H) : ATB = T\} = \{T \in B(H) : ST = T\},$$

(1.1)

where $S = L_AR_B$.

An operator $C \in B(H)$ is called positive, if $\langle Cx, x \rangle \geq 0$ for all $x \in H$. Then for any positive operator $C \in B(H)$ we define $\text{tr}C = \sum_{n=1}^\infty \langle e_n, Ce_n \rangle$. The number $\text{tr}C$ is called the trace of $C$ and is independent of the orthonormal basis chosen. An operator $C \in B(H)$ is called trace class if and only if $\text{tr}|C| < \infty$ for $|C| = (C^*C)^{1/2}$, where $C^*$ is adjoint of $C$. The family of all trace class operators is denoted by $L_1(H)$. The basic properties of $L_1(H)$ and the functional $\text{tr}(\cdot)$ are the following:

(i) Let $\|\cdot\|_1$ be defined in $L_1(H)$ by $\|C\|_1 = \text{tr}|C|$. Then $L_1(H)$ is a Banach space with the norm $\|\cdot\|_1$ and $\|C\| \leq \|C\|_1$.

(ii) $L_1(H)$ is *- ideal, that is,

(a) $L_1(H)$ is a linear space,

(b) if $C \in L_1(H)$ and $D \in B(H)$, then $CD \in L_1(H)$ and $DC \in L_1(H)$,

(c) if $C \in L_1(H)$, then $C^* \in L_1(H)$.

(iii) $\text{tr}(\cdot)$ is linear.

(iv) $\text{tr}(CD) = \text{tr}(DC)$ if $C \in L_1(H)$ and $D \in B(H)$.

(v) $B(H) = L_1(H)^*$, that is, the map $T \rightarrow \text{tr}(T)$ is an isometric isomorphism of $B(H)$ onto $L_1(H)^*$, (see [3]).

Let $X$ be a Banach space. If $M \subset X$, then

$$M^\perp = \{x^* \in X^*: \langle x, x^* \rangle = 0, \ x \in M\}$$

(1.2)

is called the annihilator of $M$. If $N \subset X^*$, then

$$^\perp N = \{x \in X : \langle x, x^* \rangle = 0, \ x^* \in N\}$$

(1.3)
is called the preannihilator of $N$. Rudin [4] proved for these subspaces:
(i) $(\bot(M^\perp))^\perp$ is the norm closure of $M$ in $X$.
(ii) $(\bot N)^\perp$ is the weak-$*$ closure of $N$ in $X^*$.

2. Main results

Lemma 2.1. Let $X$ be a Banach space. If $P$ is a continuous operator in the weak-$*$ topology on the dual space $X^*$, then there exists an operator $T$ on $X$ such that $P = T^*$.

Proof. If $P : X^* \to X^*$, then $P^* : X^{**} \to X^{**}$. We know that the continuous functionals in the weak-$*$ topology on $X^*$ are simply elements of $X$, (see [4]). Then we must show that $P^*x$ is continuous in the weak-$*$ topology on $X^*$ for all $x \in X$. Let $(x'_n)$ be a sequence in $X^*$ such that $x'_n \to x'$, $x' \in X^*$. Then we have

$$
\langle P^*x, x'_n \rangle = \langle x, Px'_n \rangle \to \langle x, Px' \rangle = \langle P^*x, x' \rangle.
$$

(Hence $P^*x$ is continuous in the weak-$*$ topology on $X^*$ for all $x \in X$, so $P^*x \in X$. If $T$ is the restriction to $X$ of $P^*$, then we have

$$
\langle x, T^*_x x' \rangle = \langle Tx, x' \rangle = \langle P^*x, x' \rangle = \langle x, Px' \rangle
$$

for all $x \in X$ and $x' \in X^*$. Hence $P = T^*$.)

Definition 2.2. If $P_\phi$ is the operator $T$ in Lemma 2.1, then $P_\phi$ is called the preadjoint operator of $P$.

The operator $x \otimes y \in B(H)$ for each $x, y \in H$ is defined by $(x \otimes y)z = \langle z, y \rangle x$ for all $z \in H$. It is easy to see that this operator has the following properties:

(i) $T(x \otimes y) = Tx \otimes y$.

(ii) $(x \otimes y)T = x \otimes T^*y$.

(iii) $\text{tr}(x \otimes y) = \langle y, x \rangle$.

The following lemma is an easy application of some properties of the operator $x \otimes y$ ($x, y \in H$) and the functional $\text{tr}(\cdot)$.

Lemma 2.3. (i) Suppose $K$ is a closed subset in the weak-$*$ topology of $B(H)$. Then $K$ is closed in the weak-$*$ topology of $B(H)$.

(ii) $S = L_AR_B$ is continuous in the weak-$*$ topology of $B(H)$ for all $A, B \in B(H)$, satisfying $\|A\| \leq 1$ and $\|B\| \leq 1$.

Lemma 2.4. There exists a linear subspace $M$ of $L_1(H)$ such that $\Delta(H) = M^\perp$ and $M$ is closed linear span of $\{S_nX \setminus X : X \in L_1(H)\}$, where $S_n$ is the preadjoint operator of $S$.

Proof. Note that

$$
\bot \Delta(A, B) = \{U \in L_1(H) : \langle U, U^* \rangle = 0, U^* \in \Delta(A, B)\}.
$$

It is known that $(\bot \Delta(A, B))^\perp$ is the weak-$*$ closure of $\Delta(A, B)$ (see [4]). Then we can write $(\bot \Delta(A, B))^\perp = \Delta(A, B)$, since $\Delta(A, B)$ is a closed set in the weak-$*$ topology of $B(H)$. We say $\bot \Delta(A, B) = M$. Now we show that $M$ is the closed linear span of $\{S_nU \setminus U : U \in L_1(H)\}$. For this, it is sufficient to prove that $\langle S_nU \setminus U, T \rangle = 0$ for all $T \in \Delta(A, B)$. 

Indeed since $ST = T$, we have
\[ \langle S_x X - X, T \rangle = \langle (S_x - I) X, T \rangle = \langle X, (S_x - I)^* T \rangle = \langle X, (S - I) T \rangle = 0. \] (2.4)

**Lemma 2.5.** Let $K(T)$ be the closed convex hull of $\{S^n T : n = 1, 2, \ldots\}$ in the weak operator topology, for a fixed $T \in B(H)$. Then we have
\[ K(T) \cap \Delta(A,B) \neq 0. \] (2.5)

**Proof.** Assume $K(T) \cap \Delta(A,B) = 0$. By Lemma 2.3, $K(T)$ is closed in the weak-* topology. It is easy to see that $K(T)$ is bounded. Then $K(T)$ is compact in the weak-* topology by Alaoglu, [1]. Since $S$ is continuous in the weak-* topology, if $U_{\alpha} \to U$ for $(U_{\alpha})_{\alpha \in I} \subset \Delta(A,B)$, then $SU_{\alpha} \to SU$. Hence $\Delta(A,B)$ is closed in the weak-* topology. This shows that $U \in \Delta(A,B)$.

Since $K(T)$ is compact and convex in the weak-* topology, and $\Delta(A,B)$ is closed in the weak-* topology, and $K(T) \cap \Delta(A,B) = 0$, there exist some $U_0 \in M$ and $\sigma > 0$ such that
\[ |\text{tr}(TU_0)| \geq \sigma \] (2.6)
for all $T \in \Delta(A,B)$, (see [2]). Now we define the operators $T_n \sum_{k=1}^n S^k T$ for all positive integer $n$. These operators are clearly in $K(T)$. It is easy to show that the operators $T_n$ is bounded. Also by Lemma 2.4, there is a $U \in L_1(H)$ such that $U_0 = S_x U - U$. Then we have
\[ |\langle T_n, U_0 \rangle| = |\langle T_n, S_x U - U \rangle| = |\langle ST_n, U \rangle - \langle T_n, U \rangle| \]
\[ = \left| \left\langle S \left( \frac{1}{n} \sum_{k=1}^n A^k TB^k \right), U \right\rangle - \left\langle \frac{1}{n} \sum_{k=1}^n A^k TB^k, U \right\rangle \right| \]
\[ = \left| \left\langle \frac{1}{n} \sum_{k=1}^n A^k TB^k, U \right\rangle - \left\langle \frac{1}{n} \sum_{k=1}^n A^{k+1} TB^{k+1}, U \right\rangle \right| \]
\[ = \frac{1}{n} \left| \left\langle A^n + TB^n - ATB, U \right\rangle \right| \]
\[ \leq \frac{1}{n} 2^n \|T\| \cdot \|U\|. \] (2.7)
This implies that $|\langle T_n, X_0 \rangle| \to 0$, which is a glaring contradiction to (2.6). 

**Theorem 2.6.** Let $H$ be separable Hilbert space and $T \in B(H)$. Then we have
\[ (i) \quad d(T, \Delta(A,B)) \geq (1/2) \sup_n \|S^n T - T\|, \]
\[ (ii) \quad d(T, \Delta(A,B)) \leq \sup_n \|S^n T - T\|. \]

**Proof.** (i) We can write
\[ S^n T - T = S^n (T - T_0) - (T - T_0) + S^n T_0 - T_0 \] (2.8)
for each $T_0 \in \Delta(A,B)$. Hence we have
\[ \|S^n T - T\| \leq \|S^n T - T_0\| + \|T - T_0\| \leq 2 \|T - T_0\|. \] (2.9)
This shows that
\[ \frac{1}{2} \sup_n \|S^nT - T\| \leq \inf_{T_0 \in \Delta(A,B)} \|T - T_0\|. \]  
(2.10)

The inequality (2.10) gives that
\[ d(T, \Delta(A,B)) \geq \frac{1}{2} \sup_n \|S^nT - T\|. \]  
(2.11)

(ii) Let \( K(T) \) be as Lemma 2.5. Then we can write
\[ K(T) = \text{co}\{S^nT : n = 1, 2, \ldots\}. \]  
(2.12)

Now take any element \( U = \sum_{k=1}^{n} \lambda_k S^kT \) in the set \( \text{co}\{S^nT : n = 1, 2, \ldots\} \), where \( \sum_{k=1}^{n} \lambda_k = 1, \lambda_k \geq 0. \) Then
\[ \|U - T\| = \left\| \sum_{k=1}^{n} \lambda_k S^kT - T \right\| 
\leq \left\| \sum_{k=1}^{n} \lambda_k S^kT - \sum_{k=1}^{n} \lambda_k T \right\| 
\leq \sum_{k=1}^{n} \lambda_k \|S^kT - T\| 
\leq \sum_{k=1}^{n} \lambda_k \sigma(T) = \sigma(T), \]  
(2.13)

where \( \sigma(T) = \sup_n \|S^nT - T\| \). That is, for all \( U \in \text{co}\{S^nT : n = 1, 2, \ldots\} \) is
\[ \|U - T\| \leq \sup_n \|S^nT - T\|. \]  
(2.14)

Since there is a sequence \( (U_n) \) in \( \text{co}\{S^nT : n = 1, 2, \ldots\} \) such that \( U_n \to V \) for all \( V \in K(T) \), then we write
\[ \|V - T\| \leq \|V - T_n\| + \|T_n - T\|. \]  
(2.15)

If we use the inequalities (2.14) and (2.15), we easily see that
\[ \|V - T\| \leq \sup_n \|S^nT - T\|. \]  
(2.16)

Also since \( K(T) \cap \Delta(A,B) \neq 0 \) by Lemma 2.5, then we obtain
\[ \|T - T_0\| \leq \sup_n \|S^nT - T\| \]  
(2.17)

for a \( T_0 \in K(T) \cap \Delta(A,B) \). Hence we can write
\[ d(T, \Delta(A,B)) = \inf_{U \in \Delta(A,B)} \|T - U\| \leq \|T - T_0\| \leq \sup_n \|S^nT - T\|. \]  
(2.18)

This completes the proof. \( \square \)

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