ASYMPTOTIC BEHAVIOR OF ORTHOGONAL POLYNOMIALS CORRESPONDING TO A MEASURE WITH INFINITE DISCRETE PART OFF AN ARC

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(Received 23 October 2000)

ABSTRACT. We study the asymptotic behavior of orthogonal polynomials. The measure is concentrated on a complex rectifiable arc and has an infinity of masses in the region exterior to the arc.

2000 Mathematics Subject Classification. 42C05, 30E15, 30E10.

1. Introduction. Kaliaguine has studied in [3] the asymptotic behavior of orthogonal polynomials associated to a measure of the type $\sigma_l = \alpha + \gamma_l$, where α is concentrated on a complex rectifiable arc E and is absolutely continuous with respect to the Lebesgue measure $|d\xi|$ on the arc, and γ_l is a finite discrete measure with masses A_k at the points $z_k \in \text{Ext}(E)$, $k = 1, 2, \dots, l$, that is, $\gamma_l = \sum_{k=1}^l A_k \delta_{z_k}$, $A_k > 0$, where δ_{z_k} being the Dirac measure at the points z_k . In this paper, we generalize the previous study, when $\sigma = \alpha + \gamma$, where α possess the same properties as in [3] and γ is concentrated on an infinite discrete part $\{z_k\}_{k=1}^\infty \in \text{Ext}(E)$, $\gamma = \sum_{k=1}^\infty A_k \delta_{z_k}$. The masses $\{A_k\}_{k=1}^\infty$ satisfy

$$A_k > 0, \quad \sum_{k=1}^\infty A_k < \infty. \quad (1.1)$$

We note that the cases of a closed curve and a circle studied in [4, 5] are different from the case of an arc with respect to the asymptotics of orthogonal polynomials.

2. The space $H^2(\Omega, \rho)$. Suppose that E is a rectifiable arc in the complex plane, $\Omega = \text{Ext}(E)$, $G = \{w \in \mathbb{C} / |w| > 1\}$ ($\infty \in \Omega$, $\infty \in G$), and $1/C(E) = \lim_{z \rightarrow \infty} (\Phi(z)/z) > 0$, where $\Phi: \Omega \rightarrow G$ is the conformal mapping. We denote by Ψ the inverse of Φ .

Let $\rho(\xi)$ be an integrable nonnegative function on E . If the weight function $\rho(\xi)$ satisfies the Szegő condition

$$\int_E \log(\rho(\xi)) |\Phi'(\zeta)| |d\xi| > -\infty. \quad (2.1)$$

Then one can construct the so-called Szegő function $D(z)$ associated with the domain Ω and the weight function $\rho(\xi)$ with the following properties.

$D(z)$ is analytic in Ω ; $D(z) \neq 0$ in Ω ; $D(\infty) > 0$; $D(z)$ has boundary values on both sides of E (a.e.) and $|D_+|^{-2} |\Phi'_\pm| = \rho(\xi)$ (a.e. on E).

Let $f(z)$ be an analytic function in Ω , we say that $f(z) \in H^2(\Omega, \rho)$ if and only if $f(\Psi(w))/D(\Psi(w)) \in H^2(G)$, and for a function F analytic in G , $F \in H^2(G)$ if and only if

$F(1/w) \in H^2(D)$; $w \in D$; $D = \{z \in \mathbb{C} / |z| < 1\}$. The space $H^2(D)$ is well known (see [6]). Any function from $H^2(\Omega, \rho)$ has boundary values f_+ , f_- on both sides of E , $f_+, f_- \in L^2(\rho)$. We define the norm in Hardy space by

$$\|f\|_{H^2(\Omega, \rho)} = \oint_E |f(\xi)|^2 \rho(\xi) |d\xi|. \quad (2.2)$$

Here, we take the integral on both sides of E .

3. Extremal properties of the orthogonal polynomials. We denote by P_n the set of polynomials of degree almost n . Define $\mu(\rho)$, $\mu^*(\rho)$, $m_n(\rho)$, $m_n(\sigma_l)$, and $m_n(\sigma)$ as the extremal values of the following problems:

$$\mu(\rho) = \inf \left\{ \|\varphi\|_{H^2(\Omega, \rho)}^2 : \varphi \in H^2(\Omega, \rho), \varphi(\infty) = 1 \right\}, \quad (3.1)$$

$$\mu^*(\rho) = \inf \left\{ \|\varphi\|_{H^2(\Omega, \rho)}^2 : \varphi \in H^2(\Omega, \rho), \varphi(\infty) = 1, \varphi(z_k) = 0, k = 1, 2, \dots \right\}, \quad (3.2)$$

$$m_n(\rho) = \min \left\{ \int_E |Q_n(\xi)|^2 \rho(\xi) |d\xi|, Q_n(z) = z^n + \dots \right\}, \quad (3.3)$$

$$m_n(\sigma_l) = \min \left\{ \int_E |Q_n(\xi)|^2 \rho(\xi) |d\xi| + \sum_{k=1}^l A_k |Q_n(z_k)|^2, Q_n(z) = z^n + \dots \right\}, \quad (3.4)$$

$$m_n(\sigma) = \min \left\{ \int_E |Q_n(\xi)|^2 \rho(\xi) |d\xi| + \sum_{k=1}^{\infty} A_k |Q_n(z_k)|^2, Q_n(z) = z^n + \dots \right\}. \quad (3.5)$$

We denote, respectively, by φ^* and ψ^* the extremal functions of the problems (3.1) and (3.2). We denote by $\{T_n^l(z)\}$ and $\{T_n(z)\}$ the systems of the monic orthogonal polynomials, respectively, associated to the measures σ_l and σ , that is,

$$\begin{aligned} T_n^l(z) &= z^n + \dots, \\ \int_E T_n^l(\xi) \bar{\xi}^p \rho(\xi) |d\xi| + \sum_{k=1}^l A_k T_n^l(z_k) \bar{\xi}_k^p &= 0; \quad p = 0, 1, 2, \dots, n-1, \\ T_n(z) &= z^n + \dots, \\ \int_E T_n(\xi) \bar{\xi}^p \rho(\xi) |d\xi| + \sum_{k=1}^{\infty} A_k T_n(z_k) \bar{\xi}_k^p &= 0; \quad p = 0, 1, 2, \dots, n-1. \end{aligned} \quad (3.6)$$

It is easy to see that the polynomials $\{T_n^l(z)\}$ and $\{T_n(z)\}$ are, respectively, the optimal solutions of the extremal problems (3.4) and (3.5).

LEMMA 3.1. *Let $\varphi \in H^2(\Omega, \rho)$ such that $\varphi(\infty) = 1$ and $\varphi(z_k) = 0$, $k = 1, 2, \dots$, and let*

$$B(z) = \prod_{k=1}^{\infty} \frac{\Phi(z) - \Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)} - 1} \frac{|\Phi(z_k)|^2}{\Phi(z_k)} \quad (3.7)$$

be the Blaschke product, then

- (1) $B \in H^2(\Omega, \rho)$; $B(\infty) = 1$; $|B_{\pm}(\xi)| = \prod_{k=1}^{\infty} |\Phi(z_k)|$ (a.e. on E),
 (2) $\varphi/B \in H^2(\Omega, \rho)$ and $(\varphi/B)(\infty) = 1$.

The proof is the same as that of Lemma 3.1 given in [1].

LEMMA 3.2. *The extremal functions φ^* and ψ^* are connected by*

$$\psi^* = B(z) \cdot \varphi^*, \quad \mu^*(\rho) = \left(\prod_{k=1}^{\infty} |\Phi(z_k)| \right)^2 \mu(\rho). \quad (3.8)$$

The proof is the same as that of a closed curve given in [2, Lemma 4.2]. We replace the finite Blaschke product by the infinite product B and using its properties announced by Lemma 3.1.

4. Main results

DEFINITION 4.1. The measure $\sigma = \alpha + \gamma$ belongs to the class A (and we write $\sigma \in A$), if the absolutely continuous part α and the discrete part of σ satisfy (in addition to conditions (1.1) and (2.1))

$$\left(\sum_{k=1}^{\infty} |\Phi(z_k)| - 1 \right) < \infty. \quad (4.1)$$

An arc E is from $C^{\alpha+}$ class if E is rectifiable and its coordinates are α -times differentiable, with α th derivatives satisfying a Lipschitz condition positive exponent.

THEOREM 4.2. *Let σ be a measure, $\sigma = \alpha + \gamma$, such that $\sigma \in A$. Then*

$$\lim_{l \rightarrow \infty} m_n(\sigma_l) = m_n(\sigma). \quad (4.2)$$

PROOF. The extremal property of $T_n(z)$ implies that

$$m_n(\sigma_l) \leq \int_E |T_n(\xi)|^2 \rho(\xi) |d\xi| + \sum_{k=1}^l A_k |T_n(z_k)|^2 \leq m_n(\sigma), \quad (4.3)$$

then

$$m_n(\sigma_l) \leq m_n(\sigma). \quad (4.4)$$

On the other hand, the extremal property of $T_n(z)$ implies that

$$\begin{aligned} m_n(\sigma) &\leq \int_E |T_n^l(\xi)|^2 \rho(\xi) |d\xi| + \sum_{k=1}^{\infty} A_k |T_n^l(z_k)|^2 \\ &= m_n(\sigma_l) + \sum_{k=l+1}^{\infty} A_k |T_n^l(z_k)|^2. \end{aligned} \quad (4.5)$$

According to the reproducing property of the kernel function $K_n(\xi, z)$ (see [7]), and $T_n^l(z) \in P_n$, we have

$$T_n^l(z_k) = \int_E T_n^l(\zeta) \overline{K_n(\xi, z_k)} \rho(\xi) |d\xi|. \quad (4.6)$$

The Scharwz inequality and the fact that $|\Phi(\xi)| = 1$ for $\xi \in E$ and $K_n(z, z_k) \in P_n$ imply

$$\begin{aligned} |T_n^l(z_k)|^2 &\leq \int_E |T_n^l(\xi)|^2 \rho(\xi) |d\xi| \cdot \int_E |K_n(\xi, z_k)|^2 \rho(\xi) |d\xi| \\ &\leq m_n(\sigma_l) \cdot K_n(z_k, z_k). \end{aligned} \quad (4.7)$$

The inequalities (1.1), (4.5), and (4.7) imply

$$\begin{aligned} m_n(\sigma) &\leq m_n(\sigma_l) + \sum_{k=l+1}^{\infty} A_k m_n(\sigma_l) K_n(z_k, z_k) \\ &\leq m_n(\sigma_l) \left[1 + \sup_{k \geq l+1} K_n(z_k, z_k) \sum_{k=l+1}^{\infty} A_k \right], \end{aligned} \quad (4.8)$$

so we have

$$\frac{m_n(\sigma)}{m_n(\sigma_l)} \leq 1 + \delta_l, \quad \text{where } \delta_l \rightarrow 0, l \rightarrow \infty. \quad (4.9)$$

Using (4.4) and (4.9), we obtain

$$m_n(\sigma) \leq \liminf_{l \rightarrow \infty} m_n(\sigma_l) \leq \limsup_{l \rightarrow \infty} m_n(\sigma_l) \leq m_n(\sigma), \quad \forall n, \quad (4.10)$$

this implies that

$$\lim_{l \rightarrow \infty} m_n(\sigma_l) = m_n(\sigma), \quad \forall n. \quad (4.11)$$

□

THEOREM 4.3. *Let σ be a measure, $\sigma = \alpha + \gamma$, such that $\sigma \in A$ and*

$$\frac{m_n(\sigma_l)}{m_n(\rho)} \leq \left(\prod_{k=1}^l |\Phi(z_k)| \right)^2, \quad \forall n, \forall l. \quad (4.12)$$

Suppose that $E \in C^{2+}$. Then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{m_n(\sigma)}{C(E)^{2n}} &= \mu^*(\rho), \\ \int_E |C(E)^{-n} T_n(\xi) - H_n(\xi)|^2 \rho(\xi) |d\xi| &\rightarrow 0, \\ T_n(z) &= C(E)^n \Phi^n(z) [\psi^*(z) + \epsilon_n(z)], \end{aligned} \quad (4.13)$$

where $H_n(\xi) = \Phi_+^n(\xi) \psi_+^*(\xi) + \Phi_-^n(\xi) \psi_-^*(\xi)$, $\epsilon_n \rightarrow 0$ uniformly on the compact subsets of Ω .

PROOF. By passing to the limit when l tends to infinity and using Theorem 4.2 and (4.12), we obtain

$$\frac{m_n(\sigma)}{C(E)^{2n}} \leq \left(\prod_{k=1}^{\infty} |\Phi(z_k)| \right)^2 \frac{m_n(\rho)}{C(E)^{2n}}. \quad (4.14)$$

This implies that

$$\limsup_{n \rightarrow \infty} \frac{m_n(\sigma)}{C(E)^{2n}} \leq \left(\prod_{k=1}^{\infty} |\Phi(z_k)| \right)^2 \mu(\rho) = \mu^*(\rho) \quad (4.15)$$

(see Lemma 3.2).

The extremal property of the polynomials $T_n(z)$ and the fact that $|\Phi(\xi)| = 1$, for $\xi \in E$ imply (see [2] for details)

$$\frac{2m_n(\sigma)}{C(E)^{2n}} = \left\| \frac{T_n}{[C(E)\Phi]^n} \right\|_{H^2(\Omega, \rho)}^2 + 2 \sum_{k=1}^{\infty} A_k \left| \frac{T_n(z_k)}{[C(E)\Phi(z_k)]^n} \right| |\Phi(z_k)|^{2n}, \quad (4.16)$$

so

$$\left\| \frac{T_n}{[C(E)\Phi]^n} \right\|_{H^2(\Omega, \rho)}^2 \leq \frac{2m_n(\sigma)}{C(E)^{2n}}. \quad (4.17)$$

Equations (4.15) and (4.17) imply that

$$\limsup_{n \rightarrow \infty} \left\| \frac{T_n}{[C(E)\Phi]^n} \right\|_{H^2(\Omega, \rho)}^2 \leq 2\mu^*(\rho). \quad (4.18)$$

Now we take the integral

$$\begin{aligned} I_n &= \int_E |C(E)^{-n} T_n(\xi) - H_n(\xi)|^2 \rho(\xi) |d\xi| \\ &= \int_E \left| \left(\frac{1}{2} C(E)^{-n} T_n(\xi) - \Phi_+^n(\xi) \psi_+^*(\xi) \right) \right. \\ &\quad \left. + \left(\frac{1}{2} C(E)^{-n} T_n(\xi) - \Phi_-^n(\xi) \psi_-^*(\xi) \right) \right|^2 \rho(\xi) |d\xi|, \end{aligned} \quad (4.19)$$

by the triangular inequality, we have

$$\begin{aligned} I_n^{1/2} &\leq \left(\int_E \left| \frac{1}{2} C(E)^{-n} T_n(\xi) - \Phi_+^n(\xi) \psi_+^*(\xi) \right|^2 \rho(\xi) |d\xi| \right)^{1/2} \\ &\quad + \left(\int_E \left| \frac{1}{2} C(E)^{-n} T_n(\xi) - \Phi_-^n(\xi) \psi_-^*(\xi) \right|^2 \rho(\xi) |d\xi| \right)^{1/2} \\ &\leq 2 \left(\int_E \left| \frac{1}{2} C(E)^{-n} T_n(\xi) - \Phi^n(\xi) \psi^*(\xi) \right|^2 \rho(\xi) |d\xi| \right)^{1/2}. \end{aligned} \quad (4.20)$$

Then we deduce that

$$I_n \leq 4 \left\| \frac{1}{2} \frac{T_n}{[C(E)\Phi]^n} - \psi^* \right\|_{H^2(\Omega, \rho)}^2. \quad (4.21)$$

By using the parallelogram rule in $H^2(\Omega, \rho)$, we have

$$I_n \leq 4 \left[2 \left\| \frac{1}{2} \frac{T_n}{C(E)^n} \right\|_{H^2(\Omega, \rho)}^2 + 2 \|\psi^*\|_{H^2(\Omega, \rho)}^2 - \left\| \frac{1}{2} \frac{T_n}{C(E)^n} + \psi^* \right\|_{H^2(\Omega, \rho)}^2 \right], \quad (4.22)$$

so

$$\limsup_{n \rightarrow \infty} I_n \leq 4 \left[\mu^*(\rho) + 2\mu^*(\rho) - \frac{9}{4} \frac{4}{3} \mu^*(\rho) \right] = 0, \quad (4.23)$$

where we have used the fact that $\liminf_{n \rightarrow \infty} \|g_n\|_{H^2(\Omega, \rho)}^2 \geq 2\mu^*(\rho) \geq (4/3)\mu^*(\rho)$, since the function $g_n(z) = (2/3)((1/2)T_n(z)/C(E)^n + \psi^*(z)) \in H^2(\Omega, \rho)$, $g_n(\infty) = 1$, and $g_n(z_k) \rightarrow 0$, $n \rightarrow \infty$. This yields

$$0 \leq \liminf_{n \rightarrow \infty} I_n \leq \limsup_{n \rightarrow \infty} I_n \leq 0, \quad (4.24)$$

finally,

$$\lim_{n \rightarrow \infty} I_n = 0. \quad (4.25)$$

For the asymptotics in the region exterior to the arc E we need the Szegő reproducing kernel function $K(\xi, z)$ (see [8, page 173]) and the fact that $T_n(z)/C(E)^n \Phi^n(z) \in H^2(\Omega, \rho)$ for all $z \in \Omega$, then

$$\begin{aligned} \frac{T_n(z)}{C(E)^n \Phi^n(z)} &= \oint_E \frac{T_n(\xi)}{C(E)^n \Phi^n(\xi)} \overline{K(\xi, z)} \rho(\xi) |d\xi| \\ &= \int_E C^{-n} T_n(\xi) \left\{ \Phi_+^{-n}(\xi) \overline{K_+(\xi, z)} + \Phi_-^{-n}(\xi) \overline{K_-(\xi, z)} \right\} \rho(\xi) |d\xi| \\ &= \int_E \left\{ C^{-n} T_n(\xi) - H_n(\xi) \right\} \left\{ \Phi_+^{-n}(\xi) \overline{K_+(\xi, z)} + \Phi_-^{-n}(\xi) \overline{K_-(\xi, z)} \right\} \rho(\xi) |d\xi| \\ &\quad + \int_E H_n(\xi) \left\{ \Phi_+^{-n}(\xi) \overline{K_+(\xi, z)} + \Phi_-^{-n}(\xi) \overline{K_-(\xi, z)} \right\} \rho(\xi) |d\xi|. \end{aligned} \quad (4.26)$$

The first integral approaches 0 as $n \rightarrow \infty$ (part 2 of [Theorem 4.3](#)), the second one may be transformed into the form

$$\begin{aligned} &\int_E \left\{ \Phi_+^n(\xi) \psi_+^*(\xi) + \Phi_-^n(\xi) \psi_-^*(\xi) \right\} \left\{ \Phi_+^{-n}(\xi) \overline{K_+(\xi, z)} + \Phi_-^{-n}(\xi) \overline{K_-(\xi, z)} \right\} \rho(\xi) |d\xi| \\ &= \oint_E \psi^*(\xi) \overline{K(\xi, z)} \rho(\xi) |d\xi| \\ &\quad + \int_E \left\{ \Phi_+^n(\xi) \psi_+^*(\xi) \Phi_-^{-n}(\xi) \overline{K_-(\xi, z)} + \psi_-^*(\xi) \Phi_+^{-n}(\xi) \Phi_-^n(\xi) \overline{K_+(\xi, z)} \right\} \rho(\xi) |d\xi| \\ &= \psi^*(z) + \lambda_n, \end{aligned} \quad (4.27)$$

where $\lambda_n \rightarrow 0$ (coefficients of an integrable function). This proves part 3. $\square$

REMARK 4.4. It is not difficult to find families of points $\{A_k\}_{k=1}^\infty$ and $\{z_k\}_{k=1}^\infty$ satisfying condition (4.12). For example if $E = [-1, +1]$, then

$$\Phi(z) = z + \sqrt{z^2 - 1} \quad \left(\left| z + \sqrt{z^2 - 1} \right| > 1 \right). \quad (4.28)$$

We can take z_k such that

$$\Phi(z_k) = 1 + \frac{1}{k^2}, \quad A_k = \frac{1}{2k}. \quad (4.29)$$

As weight function we take

$$\rho(\xi) = (1 - \xi^2)^{-1/2}. \quad (4.30)$$

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