MULTIMODAL CYCLES WITH LINEAR MAP HAVING EXACTLY ONE FIXED POINT

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Abstract. We describe a class of cycles that cannot be forced by a cycle whose linear map has exactly one fixed point.

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1. Introduction. This note is concerned with the forcing relation on cycles. In particular, we consider cycles \( \theta \) for which the \( \theta \)-linear map has exactly one fixed point. We prove a theorem which describes a large class of cycles that cannot be forced by \( \theta \).

2. Definitions. Throughout this note, \( f : I \rightarrow I \) denotes a continuous map of a compact interval. For \( x \in I \), \( f^0(x) = x \), and for \( n \in \mathbb{N} \), \( f^n(x) = f(f^{n-1}(x)) \). An element \( x \in I \) is a periodic point for \( f \) if there exists \( k \in \mathbb{N} \) satisfying \( f^k(x) = x \). The least such \( k \) is called the period of \( x \). A point of period 1 is called a fixed point. The orbit of \( x \in I \) is the set \( \{ f^n(x) \}_{n=0}^{\infty} \) and is denoted \( \mathcal{O}(x) \). If \( x \) is periodic with period \( k \), then \( \mathcal{O}(x) \) is a finite set consisting of \( k \) distinct elements.

A cycle of order \( n \) is a bijection \( \theta : \{1,2,\ldots,n\} \rightarrow \{1,2,\ldots,n\} \) satisfying \( \theta^k(1) \neq 1 \) for \( 1 \leq k < n \). Let \( x \) be a periodic point for \( f \) with least period \( n \) and \( \mathcal{O}(x) = \{ x_1 < x_2 < \cdots < x_n \} \). We say that \( x \) has orbit type \( \varnothing \) if \( \varnothing \) is a cycle of order \( n \) and \( f(x_i) = x_{\theta(i)} \) for \( 1 \leq i \leq n \). In this case, we also say that the periodic orbit \( \mathcal{O}(x) \) has orbit type \( \varnothing \). We say that \( f \) has a periodic orbit of orbit type \( \varnothing \) if there exists a periodic point \( x \in I \) which has orbit type \( \varnothing \). A cycle \( \varnothing \) forces a cycle \( \eta \) if whenever \( f \) has a periodic orbit of type \( \varnothing \), \( f \) has a periodic point of type \( \eta \).

For a cycle \( \varnothing \) of order \( n \), the \( \varnothing \)-linear map \( L_\varnothing : [1,n] \rightarrow [1,n] \) is defined by

\[
L_\varnothing(k) = \varnothing(k), \quad \text{for } 1 \leq k \leq n,
\]

\[
L_\varnothing \text{ is linear on } [i,i+1], \quad \text{for } 1 \leq i \leq n-1.
\]

The graph of \( L_\varnothing \) consists of at most \( n - 1 \) linear segments, each having a slope \( m \) satisfying \( |m| \geq 1 \). A cycle \( \eta \) is forced by \( \varnothing \) if and only if \( L_\varnothing \) has a periodic orbit of type \( \eta \) [1].

Baldwin [2] defined the forcing relation and proved that the forcing relation induces a partial order on the set of cycles. He provided an exhaustive but inefficient algorithm for determining whether one cycle forces another. Jungreis [6] provided a combinatorial method to determine if one cycle forces another in certain cases. In [3] a geometric version of Jungreis’s algorithm is given and in [4] this algorithm is generalized to any
two cycles. In [8], another geometric algorithm is given to determine the forcing relation. This algorithm is similar to Baldwin’s original algorithm but more efficient. A cycle is called unimodal if \( L_\theta \) has exactly one turning point (a maximum, say). In [5] the forcing relation on the set of unimodal cycles is studied. In particular, it is shown that the forcing relation induces a total order on the set of unimodal cycles. In [7, 9] the structure of this totally ordered set is investigated.

3. Preliminaries. In this section, we define the \( RL \)-pattern for any cycle, and we define the step number for a cycle \( \theta \) for which \( L_\theta \) has exactly one fixed point.

**Definition 3.1.** Let \( \eta \) be any cycle of order \( k \). The \( RL \)-pattern for \( \eta \) is the sequence
\[
G = G_1 G_2 \cdots G_k \in \{R, L\}^k
\]
defined by
\[
G_i = \begin{cases} 
R & \text{if } \eta^i(1) > \eta^{i-1}(1), \\
L & \text{if } \eta^i(1) < \eta^{i-1}(1). 
\end{cases}
\]
Let \( R(\eta) \) denote the length of the longest string of consecutive \( R \)'s in the \( RL \)-pattern for \( \eta \).

Obviously, every \( RL \)-pattern begins with an \( R \) and ends with an \( L \).

Let \( \theta \) be a cycle of order \( n \) such that \( L_\theta \) has exactly one fixed point. Let \( p_1 \in (1, n) \) denote the unique fixed point and let \( E_1 = \{x < p_1 \mid f(x) = p_1\} \). If \( E_1 \neq \emptyset \), we let \( p_2 = \max\{E_1\} \). For \( i > 1 \), if the points \( p_1, p_2, \ldots, p_i \) and nonempty sets \( E_1, \ldots, E_{i-1} \) have been defined, we set
\[
E_i = \{x < p_i \mid f(x) = p_i\}. \tag{3.3}
\]
If \( E_i \neq \emptyset \), we let \( p_{i+1} = \max\{E_i\} \). We see that for some \( i \geq 1 \), \( E_i = \emptyset \), for otherwise, there would exist a strictly decreasing sequence \( \{p_n\}_{n=1}^\infty \) in \([1, n]\), converging to a point \( p < p_1 \) but satisfying, for each \( n \),
\[
L_\theta(p_n) = p_{n-1}, \tag{3.4}
\]
so that by continuity,
\[
\lim_{n \to \infty} L(p_n) = L(p) \tag{3.5}
\]
and at the same time
\[
\lim_{n \to \infty} L(p_n) = \lim_{n \to \infty} p_{n-1} = p. \tag{3.6}
\]
Thus \( L(p) = p \), which would contradict the assumption that \( L_\theta \) has exactly one fixed point. Therefore we can make the following definition.

**Definition 3.2.** Let \( \theta \) be a cycle of order \( n \) such that \( L_\theta \) has exactly one fixed point. The step number of \( \theta \), denoted \( S(\theta) \), is the (smallest) value of \( i \) for which \( E_i = \emptyset \).

**Example 3.3.** The cycle \( \eta_1 = (1 \ 2 \ 3 \ 4) \) has \( RL \)-pattern \( RRRL \). The cycle \( \eta_2 = (1 \ 4 \ 7 \ 2 \ 6 \ 8 \ 5) \) has \( RL \)-pattern \( RRLRLRLL \); \( R(\eta_1) = 3 \) and \( R(\eta_2) = 2 \).
4. Results. For any cycle \( \theta \) such that \( L_\theta \) has exactly one fixed point, the following theorem describes a large class of cycles that cannot be forced by \( \theta \).

**Theorem 4.1.** Let \( \theta \) be a cycle of order \( n \geq 2 \) such that \( L_\theta \) has exactly one fixed point. Let \( S(\theta) \) denote the step number of \( \theta \). Let \( \eta \) be any cycle. If \( R(\eta) > S(\theta) \), then \( \theta \) does not force \( \eta \).

**Proof.** We have

\[
1 < p_{S(\theta)} < p_{S(\theta) - 1} < \cdots < p_2 < p_1 < n. \tag{4.1}
\]

We write

\[
[1, n] = \bigcup_{i=1}^{S(\theta) + 1} I_i, \tag{4.2}
\]

where

\[
I_1 = [p_1, n],
I_i = [p_i, p_{i-1}] \quad \text{for } 2 \leq i \leq S(\theta),
I_{S(\theta) + 1} = [1, p_{S(\theta)}]. \tag{4.3}
\]

For any \( x \in \text{int}(I_1) \), \( L_\theta(x) < x \). So \( x \) cannot be the leftmost point in any periodic orbit. For \( 2 \leq i \leq S(\theta) + 1 \), we argue inductively. If \( x \in \text{int}(I_i) \), then \( L_\theta(x) \neq x \) and \( L_\theta(x) \in \bigcup_{j=i-1}^{S(\theta)} I_j \), so if \( x \) is the leftmost point of a periodic orbit of type \( \gamma \), the RL-pattern of \( \gamma \) consists of at most \( i - 1 \) consecutive \( R \)'s followed by an \( L \). That is, \( R(\gamma) \leq i - 1 \). This shows that any cycle \( \eta \) forced by \( \theta \) must have \( R(\eta) \leq S(\theta) \).

**Example 4.2.** Let \( \theta = (1 \ 2 \ 6 \ 3 \ 4 \ 5) \). \( L_\theta \) has exactly one fixed point and \( S(\theta) = 3 \). From Theorem 4.1, we know that for all \( n \geq 5 \), \( \theta \) does not force \((1 \ 2 \ 3 \ \cdots \ n)\). Using the technique developed in [8] it is seen that \( \theta \) does force \((1 \ 2 \ 3 \ 4)\) and that there are exactly two distinct orbits of type \((1 \ 2 \ 3 \ 4)\). Also, \( \theta \) forces \((1 \ 2 \ 3)\) and there are six distinct orbits of type \((1 \ 2 \ 3)\).

**Example 4.3.** Let \( \theta = (1 \ 3 \ 5 \ 2 \ 8 \ 4 \ 7 \ 6) \). \( L_\theta \) has one fixed point and \( S(\theta) = 2 \). From Theorem 4.1, we see that for all \( n \geq 4 \), \( \theta \) does not force \((1 \ 2 \ 3 \ \cdots \ n)\). Using [8], one can find exactly two distinct orbits of type \((1 \ 2 \ 4 \ 3)\), exactly fourteen distinct orbits of type \((1 \ 3 \ 2 \ 4)\), exactly eleven distinct orbits of type \((1 \ 4 \ 2 \ 3)\) and one can show that there are now orbits of type \((1 \ 3 \ 4 \ 2)\) and no orbits of type \((1 \ 4 \ 3 \ 2)\). These are the only orbit types of period 4 forced by \( \theta \).

**References**


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