Abstract. The aim of this note is to use a fixed point theorem to prove results for variational-like inequalities for pseudomonotone operators.

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1. Introduction. Recently, Singh et al. [10] studied pseudomonotone operators and derived interesting results in variational inequality and complementarity problems using a recent fixed point theorem of Tarafdar [13], which is equivalent to F-KKM theorem [13]. They derived a few interesting results as corollaries and gave an application in minimization problems. Earlier, Parida et al. [7] studied a variational-like inequality problem and developed a theory for the existence of its solution using Kakutani’s fixed point theorem, and also established the relationship between the variational-like inequality problem and some mathematical programming problems. Further results on existence theorem for variational-like inequality problems were obtained by Wadhwa and Ganguly [14] using Tarafdar’s fixed point theorem [11], which is equivalent to the KKM fixed point theorem [13].

In this note, we use Tarafdar’s result [13] and prove an existence theorem for variational-like inequality problem for $g$-pseudomonotone operators and then derive some interesting results and corollaries.

We need the following definitions:

Let $E$ stand for a real locally convex Hausdorff topological vector space and $X$ a nonempty convex subset of $E$ with $E^* \neq \{0\}$, being the continuous dual of $E$. Let $T : X \to E^*$ be a nonlinear map. The mapping $T : X \to E^*$ is hemicontinuous if $T$ is continuous from the line segment of $X$ to the weak topology of $E^*$. A point $y \in X$ is said to be a solution of the variational inequality if

$$\langle Ty, x - y \rangle \geq 0 \quad \forall x \in X. \quad (1.1)$$

Let $g$ be a continuous map, $g : X \times X \to E$. A point $y \in X$ is said to be a solution of the variational-like inequality problems if

$$\langle Ty, g(x, y) \rangle \geq 0 \quad \forall x \in X. \quad (1.2)$$

If $g(x, y) = x - y$, (1.2) reduces to (1.1) [7].

A map $T : X \to E^*$ is said to be monotone if

$$\langle Ty - Tx, y - x \rangle \geq 0 \quad \forall x, y \in X. \quad (1.3)$$
Here, \((\cdot, \cdot)\) denotes the pairing between \(E^*\) and \(E\).

The map \(T\) is called pseudomonotone if
\[
(Ty, y - x) \geq 0 \quad \text{whenever} \quad (Tx, y - x) \geq 0 \quad \forall x, y \in X.
\] (1.4)

**Definition 1.1.** A map \(T : X \to E^*\) is said to be \(g\)-monotone on \(X\) if
\[
(Tx, g(y, x)) + (Ty, g(x, y)) \leq 0 \quad \forall x, y \in X.
\] (1.5)

For \(g(y, x) = y - x\), we get the definition of monotone operators.

**Definition 1.2.** A map \(T : X \to E^*\) is said to be \(g\)-pseudomonotone if
\[
(Tx, g(y, x)) \geq 0 \quad \text{whenever} \quad (Ty, g(x, y)) \geq 0 \quad \forall x, y \in X.
\] (1.6)

For \(g(y, x) = y - x\), we get the definition of pseudomonotone operators.

We are interested in the following:

Find \(x \in X\) such that
\[
(Tx, g(y, x)) + hy \geq hx \quad \forall y \in X,
\] (1.7)

where \(T : X \to E^*\) is a nonlinear mapping and \(h : X \to \mathbb{R}\) is a low semi-continuous and convex functional.

We need the following fixed point theorem [13].

**Theorem 1.3.** Let \(X\) be a nonempty, convex subset of a Hausdorff topological vector space \(E\). Let \(F : X \to 2^X\) be a set-valued mapping such that

(i) for each \(x \in X\), \(f(x)\) is a nonempty, convex subset of \(X\);
(ii) for each \(y \in X\), \(F^{-1}(y) = \{x \in X : y \in f(x)\}\) contains a relatively open subset \(O_y\) of \(X\) (\(O_y\) may be empty for some \(y\));
(iii) \(U_{x \in X} O_x = X\); and
(iv) \(X\) contains a nonempty subset \(X_0\) contained in a compact convex subset \(X_1\) of \(X\) such that the set \(D = \bigcap_{x \in X_0} O_x^c\) is compact (\(D\) may be empty and \(O_x^c\) denotes the complement of \(O_x\) in \(X\)).

Then there exists a point \(x_0 \in X\) such that \(x_0 \in F(x_0)\).

We make the following hypothesis.

**Condition 1.4.** For \(X \subset E\), let \(T : X \to E^*\) and \(g : X \times X \to E\) satisfy the following:

(i) for each \(x \in X\), \(g(y, x)\) is convex \(y \in X\);
(ii) \(g(x, y) + g(z, x) = g(x, z)\) for all \(x, y, z \in X\);
(iii) \(g(x, x) = 0\);
(iv) for every \(x \in E^*\), \((Tx, y)\) is monotone increasing in \(y \in E^*\).

2. Main results. First, we give the following result.

**Lemma 2.1.** If \(X\) is a nonempty convex subset of a topological vector space \(E\) and \(T : X \to E^*\) is a \(g\)-pseudomonotone and hemicontinuous, then \(x \in X\) is a solution of
\[
(Tx, g(y, x)) + hy \geq hx \quad \forall y \in X
\] (2.1)
if and only if $x \in X$ is a solution of
\begin{equation}
\langle Ty, g(y, x) \rangle + hy - hx \geq 0 \quad \forall y \in X,
\end{equation}
where $h : X \rightarrow \mathbb{R}$ is a convex function and $g : X \times X \rightarrow E$ is such that it satisfies Condition 1.4.

**Proof.** Let $x \in X$ be a solution of (2.1). Then, by Condition 1.4(i), (ii) and the $g$-pseudomonotonicity of $T$, we have
\begin{equation}
\langle Ty, g(y, x) \rangle + hy - hx \geq 0 \quad \forall y \in X.
\end{equation}
Now, assume that $x$ satisfies (2.2) and let $y \in X$ be arbitrary. Then, using Minty’s technique [5],
\begin{equation}
yt = (1 - t)x + ty \in X \quad \forall t \in (0, 1)
\end{equation}
since $X$ is convex. Hence, we have
\begin{equation}
\langle Ty_t, g(y_t, x) \rangle + hy_t - hx \geq 0.
\end{equation}
So, by Condition 1.4(ii), (iii),
\begin{equation}
t \langle Ty_t, g(y, x) \rangle + t (hy - hx) \geq 0
\end{equation}
since $T$ is hemicontinuous. Letting $t \rightarrow 0$, we get
\begin{equation}
\langle Tx, g(y, x) \rangle + hy - hx \geq 0.
\end{equation}

Now, we state the following result.

**Theorem 2.2.** Let $X$ be a nonempty closed convex subset of a real Hausdorff topological vector space $E$ with $E^* \neq \{0\}$. Let $T : X \rightarrow E^*$ be a $g$-pseudomonotone and hemicontinuous map such that Condition 1.4 is satisfied, and $h : X \rightarrow \mathbb{R}$ is a lower semicontinuous and convex function. Further, assume that there exists a nonempty set $X_0$ contained in a compact convex subset $X_1$ of $X$ such that the set
\begin{equation}
D = \bigcap_{x \in X_0} \{ y \in X : \langle Tx, g(x, y) \rangle + hx - hy \geq 0 \}
\end{equation}
is either empty or compact.

Then, there exists an $x_0 \in X$ such that
\begin{equation}
\langle Tx_0, g(y, x_0) \rangle + hy - hx_0 \geq 0 \quad \forall y \in X.
\end{equation}

**Proof.** Suppose that, for each $y \in X$, there exists an $x \in X$ such that
\begin{equation}
\langle Tx, g(y, x) \rangle + hx - hy < 0.
\end{equation}
First, suppose that (2.10) does not hold. This means that there exists at least one \( y_0 \in X \) such that
\[
\langle Tx, g(y_0, x) \rangle + hx - hy_0 \geq 0 \quad \forall x \geq X,
\]
that is, \( y_0 \geq X \) is a solution of (2.2). Then, by Lemma 2.1, \( y_0 \in X \) is a solution of (2.1).

Next, assume that there is no solution of (2.1) under condition (2.10) given that (2.10) holds. Then, for each \( x \in X \), the set
\[
F(x) = \{ y \in X : \langle Tx, g(y, x) \rangle + hx - hy < 0 \}
\]
must be nonempty. It also follows from the convexity of \( h \) and by Condition 1.4 that the set \( F(x) \) is convex for each \( x \in X \). Thus, \( F : X \to 2^X \) is a set-valued map with \( F(x) \) nonempty and convex for each \( x \in X \).

Now, for each \( x \in X \),
\[
F^{-1}(x) = \{ y \in X : x \in (y) \} = \{ y \in X : \langle Ty, g(x, y) \rangle + hx - hy < 0 \}. \tag{2.13}
\]
For each \( x \in X \),
\[
\{F^{-1}(x)\}^c = \text{complement of } F^{-1}(x) \text{ in } X
\]
\[
= \{ y \in X : \langle Ty, g(x, y) \rangle + hx - hy \geq 0 \} \tag{2.14}
\]
\subset \{ y \in X : \langle Tx, g(x, y) \rangle + hx - hy \geq 0 \}
by the \( g \)-pseudomonotonicity of \( T = G(x) \).

Again, using Condition 1.4 and the convexity of \( h \), we can show that \( G(x) \) is convex for each \( x \in X \). Since \( g \) is continuous and \( h \) is lower semi-continuous, \( G(x) \) is a relatively closed subset of \( X \).

Hence, for each \( x \in X \),
\[
F^{-1}(x) \supset [G(x)]^c = 0_x \quad \text{is a relatively open subset of } X. \tag{2.15}
\]
Now, by condition (2.10), we can easily see that \( \bigcup_{x \in X} O_x = X \). (Indeed, if \( y \in X \), by (2.10), there exists an \( x \in X \) such that \( y \in [G(x)]^c = O_x \). Thus, \( y \in \bigcup_{x \in X} O_x \). Hence, \( \bigcup_{x \in X} O_x = X \).)

Finally, \( D = \bigcap_{x \in X} G(x) = \bigcap_{x \in X} O_x^c \) is compact or empty by the given condition. Hence, by Theorem 1.3, there exists an \( x \in X \) such that \( \langle Tx, g(x, x) \rangle + hx - hx < 0 \), which is impossible. Hence, there is a solution in this case as well.

Here, we give a few results that are special cases of Theorem 2.2.

**Corollary 2.3.** Let \( T : X \to E^* \) be \( g \)-monotone and hemi-continuous, where \( g \) satisfies Condition 1.4, \( h : X \to \mathbb{R} \) is convex and lower semi-continuous. Further, assume that there exists a nonempty set \( X_0 \) contained in a compact convex subset \( X_1 \) of \( X \) such that \( D = \bigcap_{x \in X_0} \{ y \in X : \langle Tx, g(x, y) \rangle + hx - hy \geq 0 \} \) is either empty or compact. Then there is an \( x \in X \) satisfying (2.1).

**Remark 2.4.** For \( g(x, y) = x - y \), Corollary 2.3 implies Corollary 1.2 of Singh et al. [10] which, in turn, implies a well-known result of Tarafdar [12].
Corollary 2.5. Let $X$ be a compact convex subset of $E$ and $T : X \to E^*$ be $g$-
pseudomonotone and hemicontinuous where $g$ satisfies Condition 1.4. Suppose that $h : X \to \mathbb{R}$ is lower semicontinuous and convex. Then there is an $x \in X$ satisfying (2.1).

Remark 2.6. For $g(x,y) = x - y$,

(i) Corollary 2.5 implies [10, Corollary 1.3].

(ii) If we take $T = A - B$, where $A$ is a monotone map and $B$ is antimonotone and both are hemicontinuous, then we derive a result due to Siddiqui et al. [8]. Here, we need only two conditions, the lower semicontinuity, and the convexity of the function $h$.

Remark 2.7. For $h = 0$, Corollary 2.5 implies Theorem 2 and Corollary 1 of Wadhwa and Ganguly [14] which implies, respectively, Theorem 2 and Corollary of Tarafdar [11]. Tarafdar’s result covered the result of Browder [1] and Theorem 1.1 of Hartman and Stampacchia [3].

Now, we prove a result similar to Theorem 2.1 of Singh et al. [9]. For $A \subset E$, $\text{int}(A)$ and $\partial(A)$ denote, respectively, the interior and the boundary of $A$, while for $A, X \subset E$, $\text{int}_X(A)$ and $\partial(A)$ denote, respectively, the relative interior and the relative boundary of $A$ in $X$. A subset of a Banach space is said to be solid if it has a nonempty interior.

Theorem 2.8. Let $X$ be a closed convex subset of a reflexive Banach space $E$ and $T : X \to E^*$ a $g$-
pseudomonotone and hemicontinuous mapping, $g : X \times X \to E$ satisfy Condition 1.4, and $h$ is convex and lower semicontinuous. Then the following conditions are equivalent:

(i) There exists $\bar{x} \in X$ such that $\langle Tx, g(x, \bar{x}) \rangle + hx - h\bar{x} \geq 0$ for all $x \in X$, that is, $x$ is a solution of (2.1).

(ii) There exists a $u \in X$ and a constant $r > \|u\|$ such that $X \langle T(x), g(x, u) \rangle + hx - hu \geq 0$ for all $x \in X$ with $\|x\| = r$.

(iii) There exists $r > 0$ such that the set $\{x \in X : \|x\| \leq r\}$ is nonempty with the property that, for each $x \in X$ with $\|x\| = r$, there exists a $u \in X$ with $\|u\| < r$ and $\langle T(x), g(x, u) \rangle hx hu \geq 0$.

Proof. This can be proved following Cottle and Yao [2, Theorem 2.2] as well as Parida et al. [7, Theorem 3.4].

Remark 2.9. For a monotone $T$ operator and $h = 0$:

(1) Theorem 2.8(i), (ii), and (iii) were obtained by Parida et al. [7].

(2) For $g(x, \hat{x}) = x - \hat{x}$, Theorem 2.8(ii) and (iii) reduce to the results of Theorems 2.3 and 2.4 of Moré [6], respectively.

Remark 2.10. For $g(x,x) = x - \hat{x}$ and $h = 0$, Theorem 2.8(i), (ii), and (iii) were obtained as Theorem 2.1(i), (ii), and (iii) by Singh et al. [9] and, in Hilbert spaces, similar results were obtained by Cottle and Yao (see [1, Theorem 2.2]).

Let $H, K$ be nonempty, closed subsets of $\mathbb{R}^n$, then we denote, by $B_H(K)$, the set of $z \in K$ such that $U(z) \cap (H - K) \neq \Phi$ and, by $I_H(K)$, the set of $z \in K$ such that $U(z) \cap (H - K) = \Phi$, for some neighbourhood $U(z)$ of $z$. 

Finally, we present a result similar to Hirano and Takahashi [4] for unbounded subsets in $\mathbb{R}^n$. Before that, we present the following result of Singh et al. [9, Corollary 1.12].

**Corollary 2.11.** Let $X$ be a closed bounded convex subset of a reflexive Banach space $E$ and $T : X \rightarrow E^*$ a pseudomonotone and hemicontinuous mapping. Then the set of solutions of variational inequality for a point $x_0 \in X$, $(Tx_0, y - x_0) \geq 0$ for all $y \in X; y \in x$; is a nonempty weakly compact convex subset of $X$.

**Theorem 2.12.** Let $X$ be a nonempty closed convex subset of $\mathbb{R}^n$ and $T : X \rightarrow \mathbb{R}^n$ be $g$-pseudomonotone such that Condition 1.4 is satisfied; $h : X \rightarrow \mathbb{R}$ a lower semicontinuous and convex function. Then there exists a solution of (2.1) in $X$ if and only if there exists a bounded closed convex subset $K$ of $X$ such that, for each $z \in B_X(K)$, there exists $y \in I_X(K)$ such that

$$
\langle Tz, g(y^*, z) \rangle + hz - hy \rightarrow 0.
$$

**(2.16)**

**Proof.** Using Corollary 2.11, with little modification, it can be shown that if there exists a solution of (2.1), then there exists a weakly compact convex subset $K$ of $X$ such that (2.16) is satisfied. Conversely, let $K$ be a weakly compact convex subset and there exists $x^* \in K$ such that

$$
\langle Tx^*, g(x, x^*) \rangle \geq 0 \quad \forall x \geq K,
$$

**(2.17)**

where $T$ is a $g$-pseudomonotone operator. The rest of the proof is similar to that of Theorem 3 of Wadhwa and Ganguly [14].

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**References**


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