FIXED POINTS VIA A GENERALIZED LOCAL COMMUTATIVITY

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Abstract. Let $g : X \to X$. The concept of a semigroup of maps which is “nearly commutative at $g$” is introduced. We thereby obtain new fixed point theorems for functions with bounded orbit(s) which generalize a recent theorem by Huang and Hong, and results by Jachymski, Jungck, Ohta, and Nikaido, Rhoades and Watson, and others.

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1. Introduction. By a semi-group of maps we mean a family $H$ of self maps of a set $X$ which is closed with respect to composition of maps $(f \circ g = fg)$ and includes the identity map $i_d(x) = x$, for $x \in X$. We often associate with a function $g : X \to X$ following semi-groups:

$$O_g = \{ g^n | n \in \mathbb{N} \cup \{0\} \},$$

(1.1)

where $\mathbb{N}$ is the set of positive integers and $g^0 = i_d$, and

$$C_g = \{ f : X \to X | fg = gf \}.$$

(1.2)

A quick check confirms that $C_g$ is a semi-group.

If $H$ is a semi-group of self maps of a set $X$ and $a \in X$, $H(a) = \{ h(a) | a \in H \}$. In particular, if $H = O_g$, $O_g(a) = \{ g^n(a) | n \in \mathbb{N} \cup \{0\} \}$ and is called the orbit of $g$ at $a$.

In general, Lemma 3.2 and some theorems in Section 3 will be stated in the context of semi-metric spaces. A semi-metric on a set $X$ is a function $d : X \times X \to [0, \infty)$ such that $d(x, y) = d(y, x)$ for $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$. A semi-metric space is a pair $(X; d)$, where $X$ is a topological space and $d$ is a semi-metric on $X$. The topology $t(d)$ on $X$ is generated by the sets $S(p, \epsilon) = \{ x | d(x, p) < \epsilon \}$ with the requirement that $p$ is an interior point of $S(p, \epsilon)$. A sequence $\{ x_n \}$ in $X$ converges in $t(d)$ to $p \in X$ (denoted as $x_n \to p$) if and only if $d(x_n, p) \to 0$. We let $t(d)$ be $T_2$ (Hausdorff) to ensure unique limits. Thus, a metric space $(X, d)$ is a semi-metric space having the triangle inequality. For further details on semi-metric spaces, see, for example, [1, 4, 6].

If $g : X \to X$, a semi-metric space $(X; d)$ is complete (g-orbitally complete) if and only if every Cauchy sequence (in the usual sense) in $X$ $(O_g(x))$ converges to a point of $X$. $g$ is continuous at $p \in X$ if and only if whenever $\{ x_n \}$ is a sequence in $X$ and $x_n \to p$, then $f(x_n) \to f(p)$. And if $S$ is a bounded subset of $X$, $\delta(S) = \sup \{ d(x, y) | x, y \in S \}$.

We are now ready to focus on the intent of this paper, namely, to introduce a generalized “local commutativity” and to demonstrate the concept’s usefulness.
2. Nearly commutative semi-groups. In [2], a semi-group \( H \) of maps is said to be *near-commutative* if and only if for each pair \( f, g \in H \), there exists \( h \in H \) such that \( fg = gh \). We generalize as follows.

**Definition 2.1.** A semi-group \( H \) of self maps of a set \( X \) is *nearly commutative* (n.c.) at \( g : X \to X \) if and only if \((f \in H)\) implies that there exists \( h \in H \) such that \( fg = gh \).

Of course, \( O_g \) and \( C_g \) are n.c. at \( g \). Observe also that a *near-commutative semi-group* \( H \) of self maps of a set \( X \) is n.c. at each \( g \in H \). The following provides for each \( a \in (0, \infty) \) an example of a semi-group \( H = S_a \) of self maps which is not near-commutative but is n.c. at a particular \( g : X \to X \).

**Example 2.2.** Let \( X = [0, \infty) \) and \( a \in (0, \infty) \). Let \( g(x) = ax \) and define

\[
S_a = \{a^n x^n \mid x \in [0, \infty), \ n \in \mathbb{N}, \ m \in \mathbb{N} \cup \{0\}\},
\]

where \( S_a \) is *nearly commutative* (n.c.) at \( g \). For if \( f(x) = a^m x^n \) is a representative element of \( S_a \), then \( fg(x) = f(g(x)) = a^{m+n} x^n \). We want \( h(x) = a^r x^s \in S_a \) such that \( fg = gh \). Now, \( g(h(x)) = a(a^r x^s) = a^{r+1} x^s \), so we can let \( s = n \) and \( r + 1 = m + n \); that is, \( r = m + (n - 1) \). Since \( n \in \mathbb{N} \) and \( (n - 1) \), \( m \in \mathbb{N} \cup \{0\} \), \( s \) and \( r \) so designated imply \( h \in S_a \). Thus, \((f \in H = S_a)\) implies that there exists \( h \in H \) such that \( fg = gh \). Since \( i_d \in S_a \), \( S_a \) is clearly a semi-group, and we are finished. On the other hand, \( S_a \) is not a *near-commutative* semi-group. For example, let \( f(x) = a^2 x^2 \) and \( h(x) = a^2 x^3 \). We want \( t(x) = a^r x^s \) such that \( fh = ht \). So we must have \( 3s = 6 \) and \( (2 + 3r) = 6 \). But then \( r = 4/3 \), and \( r \notin \mathbb{N} \cup \{0\} \).

Now, let \( M_n \) and \( N_n \) denote the set of all \( n \times n \) real matrices and the set of all nonsingular \( n \times n \) real matrices, respectively. Then, both sets \( M_n \) and \( N_n \) are semi-groups of linear transformations \( A : \mathbb{R}^n \to \mathbb{R}^n \) relative to composition of maps (matrix multiplication).

**Example 2.3.** \( N_n \) is n.c. For if \( A, B \in N_n \), there exists \( C = B^{-1}(AB) \in N_n \) such that \( AB = BC \).

**Example 2.4.** \( M_n \) is n.c. at any \( B \in N_n \), by **Example 2.3.** But \( M_n \) is not near commutative. For instance, if \( n = 2 \), \( B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), and \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \), there exists no \( 2 \times 2 \) matrix \( C \) such that \( AB = BC \).

Now, let \( g : X \to X \). Since any semi-group of self maps which commute with \( g \) is a subset of \( C_g \), we might hope that \( H_g = \{ f : X \to X \mid fg = gh \text{ for some } h : X \to X \} \) would be a maximal semi-group which is n.c. at \( g \). However, \( H_g \) so defined need not be n.c. at \( g \)! For example, let \( X = [0, \infty) \), \( g(x) = x/(x + 1) \), and \( f(x) = x/2 \). Then \( h(x) = 2x + 1 \) satisfies \( f(g(x)) = g(h(x)) \) for \( x \in [0, \infty) \). However, there exists no \( k \in H_g \) such that \( g(h(x)) = g(k(x)) \); that is, \( 2(x + 1)^{-1} + 1 = (k(x) + 1)^{-1} \) (note that \( x, k(x) \geq 0 \)).

Note that the map \( g(x) = 1/(x + 1) \) was not surjective. So consider the following example.
**Example 2.5.** Let $X$ be any set and let $g : X \to X$ be surjective. Then the family of all self mappings of $X$, $\mathcal{F} = \{ f \mid f : X \to X \}$, is n.c. at $g$. For suppose $f \in \mathcal{F}$; we need $h \in \mathcal{F}$ such that $f g(x) = g h(x)$ for all $x \in X$. So let $a \in X$. Since $g$ is onto, we can choose $x_a \in X$ such that $g(x_a) = f(g(a))$. Choose such an $x_a$ for each $a \in X$ and define $h(a) = x_a$. Then $h : X \to X$ and $g(h(a)) = g(x_a) = f(g(a))$ for $a \in X$; that is, $f g = g h$.

**Proposition 2.6.** Suppose that $H$ is a semigroup of maps which is n.c. at $g : X \to X$. If $f \in H$ and $n \in \mathbb{N}$, there exists $h_n \in H$ such that $f g^n = g^n h_n$ (i.e., $H$ is n.c. at $g^n$).

**Proof.** Let $f \in H$. Since $H$ is n.c. at $g$, there exists $h_1 \in H$ such that $f g = g h_1$. So suppose that $k \in \mathbb{N}$ such that $f g^k = g^k h_k$ for some $h_k \in H$. Then

$$f g^{k+1} = (f g^k) g = (g^k h_k) g = g^k (h_k g).$$

Since $h_k \in H$, there exists $h_{k+1} \in H$ such that $h_k g = g h_{k+1}$, and therefore (2.2) implies $f g^{k+1} = g^k (g h_{k+1}) = g^{k+1} h_{k+1}$, as desired.

Throughout this paper, $P$ denotes a function $P : [0, \infty) \to [0, \infty)$ which is non-decreasing, and satisfies $\lim_{t \to \infty} P^n(t) = 0$ for $t \in [0, \infty)$. (For example, we could let $P(t) = \alpha t$ for some $\alpha \in (0, 1)$, or $t/(t+1)$.) And throughout this paper, we appeal to the following lemma.

**Lemma 2.7.** Let $H$ be a semi-group of self maps of a set $X$ and suppose that $H$ is nearly commutative at $g : X \to X$. Let $d : X \times X \to [0, \infty)$. Suppose that for each pair $x, y \in X$ there exists a choice $r = r(\{x, y\})$, $s = s(\{x, y\}) \in H$, and $u, v \in \{x, y\}$ for which

$$d(g x, g y) \leq P(d(u, v)).$$

Then, if $n \in \mathbb{N}$, for each pair $x, y \in X$ there exist $r_n, s_n \in H$ and $u_n, v_n \in \{x, y\}$ such that

$$d(g^n x, g^n y) \leq P^n(d(r_n u_n, s_n v_n)).$$

**Proof.** By (2.3), inequality (2.4) holds for $n = 1$, so suppose that $n \in \mathbb{N}$ for which (2.4) is true. Then, if $x, y \in X$,

$$d(g^{n+1} x, g^{n+1} y) = d(g(g^n x), g(g^n y)) \leq P(d(r u, s v)), \quad (2.5)$$

where $r, s \in H$ and $u, v \in \{g^n x, g^n y\}$, by (2.3). Specifically, $u = g^n c$, $v = g^n d$, where $c, d \in \{x, y\}$. And since $r, s \in H$, there exist $r', s' \in H$ such that $r g^n = g^n r'$ and $s g^n = g^n s'$, by Proposition 2.6. So (2.4) implies that

$$d(r u, s v) = d(r g^n(c), s g^n(d)) = d(g^n(r' c), g^n(s' d)) \leq P^n(d(r_n u_n, s_n v_n)), \quad (2.6)$$

where $r_n, s_n \in H$ and $u_n, v_n \in \{r' c, s' d\}$. Thus, $r_n u_n \in \{(r_n r') c, (r_n s') d\}$, where $r_n r'$ and $r_n s'$ are elements of $H$, since $H$ is a semi-group. So $r_n u_n = r_{n+1} u_{n+1}$, where $r_{n+1} \in \{r_n r', r_n s'\}$ (i.e., $r_{n+1} \in H$) and $u_{n+1} \in \{c, d\} \subset \{x, y\}$. Similarly, $s_n v_n = s_{n+1} v_{n+1}$, where $s_{n+1} \in H$ and $v_{n+1} \in \{x, y\}$. Thus, (2.6) implies that

$$d(r u, s v) \leq P^n(d(r_{n+1} u_{n+1}, s_{n+1} v_{n+1})), \quad r_{n+1}, s_{n+1} \in H, u_{n+1}, v_{n+1} \in \{x, y\}. \quad (2.7)$$
But $P$ is nondecreasing, and therefore (2.7) and (2.5) yield
\[
d(g^{n+1}x, g^{n+1}y) \leq P^n(d(r_{n+1}u_{n+1}, s_{n+1}v_{n+1}))
= P^{n+1}(d(r_{n+1}u_{n+1}, s_{n+1}v_{n+1})),
\]
with $r_{n+1}, s_{n+1} \in H$ and $u_{n+1}, v_{n+1} \in \{x, y\}$. So, (2.4) is true for all $n$ by induction. \qed

3. Fixed point theorems

\textbf{Definition 3.1.} Let $(X; d)$ be a semi-metric space and let $H$ be a semi-group of self maps of $X$. A map $g : X \to X$ is $P$-contractive relative to $H$ if and only if (2.3) holds. (We will also say, “$g$ is a $P$-contraction relative to $H$.”)

\textbf{Lemma 3.2.} Let $(X; d)$ be a $T_2$ semi-metric space and let $H$ be a semi-group of self maps of $X$ n.c. at $g \in H$. Suppose that $g$ is $P$-contractive relative to $H$ and that $M \subset X$ such that $B = \cup \{H(c) \mid c \in M\}$ is bounded. Then $d(g^n(x), g^n(y)) \to 0$ uniformly on $B$ as $n \to \infty$. Specifically, if $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that 
\[
(n \geq k) \Rightarrow (d(g^n(x), g^n(y)) < \epsilon \forall x, y \in B).
\]

\textbf{Proof.} By hypothesis $\delta(B) < \infty$, $P^n(\delta(B)) \to 0$ as $n \to \infty$. Let $\epsilon > 0$. We can choose $k \in \mathbb{N}$ such that 
\[
P^n(\delta(B)) < \epsilon \quad \text{for } n \geq k.
\]

Let $x, y \in B$. If $n \in \mathbb{N}$, since $g$ is $P$-contractive relative to $H$, Lemma 2.7 yields $r_n, s_n \in H$ and $u_n, v_n \in \{x, y\}(\subset B)$ such that
\[
d(g^n(x), g^n(y)) \leq P^n(d(r_n u_n, s_n v_n)).
\]
Since $u_n \in B$, there exist $h \in H$ and $c \in M$ such that $u_n = h(c)$. But $r_n, h \in H$, so $r_n h \in H$. Therefore, $r_n u_n = (r_n h)(c) \in H(c) \subset B$. Likewise, $s_n v_n \in B$. But then $d(r_n u_n, s_n v_n) \leq \delta(B)$ and therefore,
\[
P^n(d(r_n u_n, s_n v_n)) \leq P^n(\delta(B)) \quad \text{for } n \in \mathbb{N},
\]
since $P$ is nondecreasing and $n$ is arbitrary. Formulae (3.2), (3.3), and (3.4) imply
\[
d(g^n(x), g^n(y)) < \epsilon \quad \text{for } n \geq k.
\]
Since the choice of $k$ in (3.2) was independent of $x$ and $y$, (3.5) holds for all $x, y \in B$. \qed

\textbf{Theorem 3.3.} Let $(X; d)$ be a $T_2$ semi-metric space, and let $H$ be a semi-group of self maps of $X$ which is n.c. at $g \in H$. Suppose that $H(a)$ is bounded for some $a \in X$ and $X$ is $g$-orbitally complete. If $g$ is a $P$-contraction relative to $H$, then $g^n(a) \to c$ for some $c \in X$. If $g$ is continuous at $c$, $g(c) = c$.

\textbf{Proof.} Since $X$ is $g$-orbitally complete, to show that $g^n(a) \to c$ for some $c \in X$ it suffices to show that $\{g^n(a)\}$ is a Cauchy sequence.
To this end, let $\epsilon > 0$. Since, $H(a)$ is bounded, Lemma 3.2 with $B = H(a)$ implies that there exists $k \in \mathbb{N}$ such that
\[
    n \geq k \Rightarrow d(g^n(a), g^n(y)) < \epsilon \quad \forall x, y \in H(a).
\] (3.6)
Therefore, if $m > n \geq k, m = n + r$ for some $r \in \mathbb{N}$, and
\[
    d(g^n(a), g^m(a)) = d(g^n(a), g^n(g^r(a))) < \epsilon,
\] (3.7)
since $a, g^r(a) \in H(a)$. We conclude that $\{g^n(a)\}$ is Cauchy, and there exists $c \in X$ such that $d(g^n(a), c) \to 0$ for any sequences $\{x_k\}$ and $\{y_k\}$ in $B$.

**Definition 3.4.** Let $X$ and $Y$ be topological spaces. A map $g : X \to Y$ is closed if and only if $g(M)$ is closed in $Y$ whenever $M$ is a closed subset of $X$.

**Theorem 3.5.** Let $(X; d)$ be a bounded and complete $T_2$ semi-metric space, and let $H$ be a semi-group of maps n.c. at some $g \in H$. Suppose that $X$ is $g$-orbitally complete and there exists $k \in \mathbb{N}$ such that for each pair $x, y \in X$, there exist $r, s \in H$ and $u, v \in \{x, y\}$ for which
\[
    d(g^k(x_k), g^k(y_k)) \leq P(d(ru, sv)).
\] (3.9)

**Proof.** Let $x \in X$. By Theorem 3.3, $\{g^n(X)\}$ converges to $p$ for some $p \in X$. Moreover, $p \in \cap \{g^n(X) \mid n \in \mathbb{N}\}$. Otherwise, there exists $k \in \mathbb{N}$ such that $p \notin g^k(X)$. Since $g^k(X)$ is closed, there exists $\epsilon > 0$ such that $S(p, \epsilon) \cap g^k(X) = \emptyset$. Thus, $d(g^n(X), p) \geq \epsilon$ for $n \geq k$ since $g^n(X)$ is a subset of $g^k(X)$ for $n \geq k$. This contradicts the fact that $g^n(x) \to p$.

In fact, $\{p\} = \cap \{g^n(X) \mid n \in \mathbb{N}\}$. For if $q \in \cap \{g^n(X) \mid n \in \mathbb{N}\}$, for each $k \in \mathbb{N}$ we can choose $x_k, y_k \in X$ such that $g^k(x_k) = p$ and $g^k(y_k) = q$. So
\[
    d(p, q) = d(g^k(x_k), g^k(y_k)) \to 0,
\] (3.8)
by Lemma 3.2 with $M = X$.

Clearly, (i) implies that $p$ is a fixed point of $g$, since $g(\{p\}) \subset \{p\}$. Thus, if $x \in X$, $d(g^n(x), p) = d(g^n(x), g^n(p)) \to 0$ as $n \to \infty$, so (iii) holds. Similarly, if $q$ is a fixed point of $g$, then $d(p, q) = (g^n(p), g^n(q)) \to 0$, so that $q = p$. Thus, $p$ is the only fixed point of $g$.

In the following we need the triangle inequality, so we require the underlying space to be a metric space.

**Theorem 3.6.** Let $(X, d)$ be a metric space and let $H$ be a semi-group of self maps of $X$ n.c. at some $g \in H$. Suppose that $X$ is $g$-orbitally complete and there exists $k \in \mathbb{N}$ such that for each pair $x, y \in X$, there exist $r, s \in H$ and $u, v \in \{x, y\}$ for which
\[
    d(g^kx, g^ky) \leq P(d(ru, sv)).
\] (3.9)
(i) If there exists \( a \in X \) such that \( H(a) \) is bounded, then there exists \( c \in X \) such that \( \lim_{n \to \infty} g^n(a) = c \). If \( h \) is continuous for some \( h \in H \), then \( h(c) = c \). (Specifically, \( g(c) = c \) if \( g \) is continuous at \( c \).)

(ii) If \( H(x) \) is bounded for each \( x \in X \), there exists a unique \( c \in X \) such that \( g^n(x) \to c \) for all \( x \in X \). If \( g \) is continuous at \( c \), \( c \) is a unique common fixed point for all \( h \in H \).

**Proof.** Suppose that \( H(a) \) is bounded. Since \( H \) is n.c. at \( g \), Proposition 2.6 says that \( H \) is n.c. at \( g^k \). And \( X \) is \( g^k \)-orbitally complete since \( X \) is \( g \)-orbitally complete. Therefore, (3.9) and Theorem 3.3 imply that

\[
\lim_{m \to \infty} (g^k)^m(a) = c \quad \text{for some } c \in X. \tag{3.10}
\]

To see that \( \lim_{n \to \infty} g^n(a) = c \), let \( \epsilon > 0 \). Then (3.10) and Lemma 3.2 (with \( B = H(a) \)) imply that there exists \( p \in \mathbb{N} \) such that \( d((g^k)^p(a), c) < \epsilon/2 \) and \( d(g^k p(x), g^k p(y)) < \epsilon/2 \) for \( x, y \in B \); that is,

\[
d(g^k p(a), c) < \frac{\epsilon}{2}, \quad d(g^k p(g^i(a)), g^k p(a)) < \frac{\epsilon}{2} \quad \forall i \in \mathbb{N}, \tag{3.11}
\]

since \( g \in H \Rightarrow g^i(a) \in H(a) \). So, if \( n > kp, n = kp + i \) for some \( i \in \mathbb{N} \), and

\[
d(g^n(a), c) \leq d(g^n(a), g^k p(a)) + d(g^k p(a), c), \tag{3.12}
\]

or

\[
d(g^n(a), c) \leq d(g^k p(g^i(a)), g^k p(a)) + d(g^k p(a), c) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \tag{3.13}
\]

by (3.11). Consequently, \( g^n(a) \to c \).

Now, let \( h \in H \) and suppose that \( h \) is continuous at \( c \). Then, \( \lim_{n \to \infty} h(g^n(a)) = h(c) \) and

\[
d(h(c), c) = \lim_{n \to \infty} d(hg^n(a), g^n(a)) = \lim_{n \to \infty} d(h(g^k)^n(a), (g^k)^n(a)). \tag{3.14}
\]

But \( H \) is n.c. at \( g^k \), so for \( n \in \mathbb{N} \) there exists \( h_n \in H \) such that \( hg^kn = g^k h_n \). Then, by (3.14),

\[
d(h(c), c) = \lim_{n \to \infty} d((g^k)^n(h_n(a)), (g^k)^n(a)) = 0, \tag{3.15}
\]

since \( a, h_n(a) \in H(a) \) and Lemma 3.2 holds for \( g^k \). Thus, (i) holds.

To prove (ii), suppose that \( H(x) \) is bounded for each \( x \in X \). If \( a, b \in X, g^n(a) \to c_a \) and \( g^n(b) \to c_b \) for some \( c_a, c_b \in X \) by (i). But \( c_a = c_b \), since \( H(a) \cup H(b) \) is bounded, and therefore, Lemma 3.2 applied to \( g^k \) implies that \( d(c_a, c_b) = \lim_{n \to \infty} d((g^k)^n(a), (g^k)^n(b)) = 0 \).

Thus, there exists a unique \( c \in X \) such that \( g^n(x) \to c \) for all \( x \in X \). We know that \( g(c) = c \) by part (i), if \( g \) is continuous at \( c \). Since \( g^n(d) = d \) for all \( n \) if \( d \) is a fixed point of \( g \), and therefore \( g^n(d) \to d, c \) must be the only fixed point of \( g \). Moreover, \( h(c) = c \) for all \( h \in H \) (even though \( h \) may not be continuous). This follows, since Proposition 2.6 applied to \( g^k \) implies that for each \( n \in \mathbb{N} \),

\[
d(c, h(c)) = d((g^k)^n(c), h(g^k)^n(c)) = d((g^k)^n(c), (g^k)^n(h_n(c))) \tag{3.16}
\]

for some \( h_n \in H \). But \( H(c) \) is bounded, so Lemma 3.2 applied to \( g^k \) implies that the right member of (3.16) converges to zero as \( n \to \infty \), and thus, \( c = h(c) \). \( \square \)
Remark 3.7. Theorem 3.3 appreciably generalizes Theorem 2.1 in [5] and Theorem 3.6 generalizes Corollary 2.3 in [5]—and hence Theorem 2 in [3] and the theorems of Rhoades and Watson [9]. Note that in Theorem 3.6(ii), the mappings \( h \in H \) \((h \neq g)\) need not be continuous. Remember also that \( C_g \) and \( O_g \) are special instances of \( H \).

The following example suggests that the requirement in Theorem 3.6(ii), that \( H(x) \) be bounded for each \( x \in X \), is not as restrictive as may first appear.

Example 3.8. Let \( S = \{ \text{continuous functions } f : [0, \infty) \rightarrow [0, \infty] \mid \text{there exists } a_f \in (0, \infty) \text{ such that } f(x) < x \text{ for } x > a_f \} \). (So, e.g., \( \{ f \mid f(x) = mx + b, \ m \in [0,1) \text{ and } b \geq 0 \} \subset S \), and \( \ln(x + b) \in S \) for \( b \geq 1 \)). Then (1) \( S \cup \{ l_d \} \) is a semi-group under composition of functions, and (2) \( O_f(x) \) is bounded for \( f \in S \) and \( x \in (0, \infty) \).

First note that, we can let \( M_f \) denote the maximum value of \( f \) on \( [0,a_f] \) for each \( f \in S \) since each \( f \) is continuous. To see that (1) is true, let \( f,g \in S \). We need only to show that \( g \circ f = gf \in S \). Clearly, \( gf \) is a continuous self map of \( [0,\infty) \). So let \( a_{gf} = \max\{a_f,M_g\} \) and suppose that \( x > a_{gf} \). We want \( gf(x) < x \). Now, \( x > a_{gf} \) implies that \( x > a_f \) so that (i) \( f(x) < x \). If \( f(x) > a_g \), then \( gf(x) < f(x) \) \( < x \) by (i) and the definition of \( a_g \). If \( f(x) \leq a_g \), \( gf(x) \leq M_g \leq a_{gf} < x \). So, in any event, \( (g \circ f)(x) < x \) if \( x > a_{gf} \), and thus, \( g \circ f \in S \). (2) follows easily by using induction to show that \( f \in S \) implies that if \( x \in [0,\infty) \), \( f^n(x) \leq \max\{x,M_f\} \) for \( n \in \mathbb{N} \). We omit the details.

If we let \( P(t) = \alpha t \) for fixed \( \alpha \in (0,1) \) and \( t \in [0,\infty) \), we have the following corollary.

Corollary 3.9. Let \( (X,d) \) be a bounded complete metric space and let \( g : X \rightarrow X \) be continuous. Suppose that \( H \) is a semi-group of self maps of \( X \) n.c. at \( g \) and \( g \in H \). If there exists \( \alpha \in (0,1) \) such that for any pair \( x,y \in X \) there exist \( r,s \in H \) and \( u,v \in [x,y] \) for which

\[
d(gx,gy) \leq \alpha d(ru,sv),
\]

then there exists a unique \( c \in X \) such that \( g^n(x) \rightarrow c \) for \( x \in X \), and \( c = gc = hc \) for all \( h \in H \).

4. Some consequences

Definition 4.1. A gauge function is an upper semicontinuous (u.s.c.) function \( \phi : [0, \infty) \rightarrow [0, \infty) \) such that \( \phi(0) = 0 \) and \( \phi(t) < t \) for all \( t > 0 \).

Lemma 4.2. Let \( (X,d) \) be a metric space and let \( H \) be a semi-group of self maps of \( X \) which is n.c. at \( g \in H \). Suppose that \( H(x,y) = H(x) \cup H(y) \) is bounded for \( x,y \in X \) and there exists a gauge function \( \phi \) such that

\[
d(gx,gy) \leq \phi(\delta(H(x,y))) \quad \text{for } x,y \in X.
\]

Then, there exists a nondecreasing continuous function \( P : [0, \infty) \rightarrow [0, \infty) \) such that \( P^n(t) \rightarrow 0 \) for all \( t > 0 \) and which satisfies the following condition: for any pair \( x,y \in X \) there exist \( r = r(x,y) \), \( s = s(x,y) \in H \), and \( u,v \in [x,y] \) such that

\[
d(gx,gy) \leq P(d(ru,sv)).
\]
Proof. Let \( x, y \in X \) and suppose that (4.1) holds. Since, \( \phi \) is a gauge function, as is well known [2], there exists a nondecreasing continuous function \( P : [0, \infty) \to [0, \infty) \) such that \( P^n(t) \to 0 \) for \( t \geq 0 \), and
\[
\phi(t) < P(t), \quad P(t) < t \quad \forall t \in (0, \infty).
\] (4.3)
Since \( P \) is continuous, (4.3) implies that for any \( t > 0 \), there exists \( \varepsilon_t \in (0, t) \) such that
\[
t' \in (t - \varepsilon_t, t + \varepsilon_t) \quad \Rightarrow \quad \phi(t) < P(t').
\] (4.4)
And since \( H(x, y) \) is bounded, the definition of \( \delta \) implies that there exist \( r, s \in H \) and \( u, v \in \{x, y\} \) such that, with \( t = \delta(H(x, y)) \),
\[
t = \delta(H(x, y)) \geq d(ru, sv) > \delta(H(x, y)) - \varepsilon_t.
\] (4.5)
So, with \( t' = d(ru, sv) \), (4.4) and (4.5) imply that
\[
\phi(\delta(H(x, y))) < P(d(ru, sv)).
\] (4.6)
Therefore, (4.1) implies that \( d(gx, gy) \leq P(d(ru, sv)). \)

The following theorem provides a generalization of Theorem 2.1 in [2].

**Theorem 4.3.** Let \( (X, d) \) be a complete metric space and let \( H \) be a semi-group of self maps of \( X \) which is n.c. at some \( g \in H \). Suppose that the following conditions are satisfied:

(i) \( H(x) \) is bounded for all \( x \in X \), \( g \) is continuous,

(ii) there exists a gauge function \( \phi \) and \( k \in \mathbb{N} \) such that \( d(g^k x, g^k y) \leq \phi(\delta(H(x, y))) \) for \( x, y \in X \).

Then

(a) \( H \) has a unique common fixed point \( c \) and \( g^n(x) \to c \) for \( x \in X \).

(b) If for each \( h \in H - \{i_d\} \) there exists \( k = k_h \in \mathbb{N} \) such that (4.7) holds with \( g = h \), then
\[
h^n(x) \to c \quad \forall x \in X, \quad h \in H - \{i_d\}.
\] (4.8)

Proof. Now, (i) implies that \( H(x, y) = H(x) \cup H(y) \) is bounded for \( x, y \in X \). To see that (a) is true, note that \( H \) is n.c. at \( g^k \) by Proposition 2.6 and substitute \( g^k \) for \( g \) in Lemma 4.2 to conclude that (3.9) holds. Consequently, we can appeal to Theorem 3.6(ii) to obtain a \( c \in X \) such that \( g^n(x) \to c \) for \( x \in X \). And since \( g \) is continuous, \( c \) is the unique fixed point of \( g \) and a fixed point for each \( h \in H \). Thus, \( c \) is the unique common fixed point of \( H \) (remember, \( g \in H \)) and therefore (a) holds.

To prove (b) note that, by part (a), if \( h \in H - \{i_d\}, \ h \neq g, \ h^n(c) = g(c) = c \) for \( n \in \mathbb{N} \). But Theorem 3.6 applied to \( h \) yields a unique \( c_1 \in X \) such that \( h^n(x) \to c_1 \) for all \( x \in X \). Since \( h^n(c) = c \) for all \( n, c_1 = c \).

Remark 4.4. Theorem 4.3 generalizes Theorem 2.1 in [2] in the following ways:

(i) The semi-group \( H \) is not required to be near-commutative (i.e., n.c. at each \( h \in H \)), but n.c. only at \( g \).
(ii) \( g \) is the only member of \( H \) required to be continuous,

(iii) in (b), (4.7) is required to hold only for \( k = k_h \), not for all \( k \geq k_h \).

**Theorem 4.3** yields the following corollary, which generalizes the theorem of Ohta and Nikaido [8] by requiring only that the orbits of \( f \)—but not all of \( X \)—be bounded.

**Corollary 4.5.** Let \( f \) be a continuous self mapping of a metric space \((X,d)\) having bounded orbits \( O_f(x) \) for all \( x \in X \). If there exist \( c \in (0,1) \) and \( k \in \mathbb{N} \) such that

\[
d(f^k x, f^k y) \leq c \delta(\{f^t | t \in \{x,y\}, i \in \mathbb{N} \})
\]

(4.9)

for all \( x, y \in X \), then \( f \) has a unique fixed point.

Observe that **Lemma 3.2** does not require that \( g \in H \), whereas the theorems in Section 3 do. The requirement that \( g \in H \) was convenient in the proof, but the following proposition says that it is not necessary when \( O_g(a) \) is bounded. Moreover, this result is needed for the proof of **Theorem 4.7**.

**Proposition 4.6.** If \( H \) is a semi-group of self maps n.c. at \( g \) and \( g \in H \), then \( H_\beta = \{g^n h | n \in \mathbb{N} \cup \{0\} \text{ and } h \in H \} \) is a semi-group which is n.c. at \( g \). Moreover, \( g \in H_\beta \) and \( H \subset H_\beta \).

**Proof.** \( H_\beta \) is a semi-group. For if \( g^n h_1, g^m h_2 \in H_\beta \), since \( H \) is n.c. at \( g \), we have

\[
g^n h_1 g^m h_2 = g^n (h_1 g^m) h_2 = g^n (g^m h_2) = g^n (h_1 h_2) = g^n h_3 h_2 \in H.
\]

\( H_\beta \) is n.c. at \( g \), since \( (H \text{ n.c. at } g) \) implies that there exists \( h_2 \in H \) such that \( (g^n h) g = g^n (h g) = g^n (g h) = g (g^n h) \).

It is clear that if \( g : X \to X \) is a \( P \)-contraction relative to \( H \), then it is certainly a \( P \)-contraction relative to \( H_\beta \) since \( H \subset H_\beta \). We use this fact in the proof of **Theorem 4.7**.

**Theorem 4.7.** Let \( C \) be a compact subset of a normed linear space \( X \) which is star-shaped with respect to \( a \in C \). Let \( T : C \to C \) be continuous and let \( H \) be a semi-group of affine maps \( I : C \to C \) n.c. at \( T \) such that \( I(q) = q \). If for each pair \( x, y \in C \) there exist \( I, J \in H \) and \( u, v \in \{x, y\} \) for which

\[
\|Tx - Ty\| \leq \|Iu - Jv\|,
\]

(4.10)

then there exists \( a \in C \) such that \( a = Ta \) and \( a = Ia \) for all continuous \( I \in H \).

**Proof.** Choose a sequence \( \{k_n\} \) in \((0,1)\) such that \( k_n \to 1 \), and for each \( n \in \mathbb{N} \), let

\[
T_n(x) = k_n Tx + (1 - k_n) q.
\]

(4.11)

Since \( C \) is star-shaped with respect to \( q \), \( T_n : C \to C \) for \( n \in \mathbb{N} \). Moreover, if \( I \in H \), there exists \( J \in H \) such that

\[
IT_n x = I(k_n Tx + (1 - k_n) q) = k_n I(Tx) + (1 - k_n) Iq
\]

\[
= k_n J(Ix) + (1 - k_n) q = T_n Jx,
\]

(4.12)

since \( I \) is affine, \( H \) is n.c. at \( T \), and \( Iq = q \). Thus, for each \( n \in \mathbb{N} \), \( H \) is a semi-group of affine maps which is n.c. at \( T_n \). Then, by **Proposition 4.6**, \( H_{T_n} \) is a semi-group of self maps of \( C \) which is n.c. at \( T_n \), \( T_n \in H_{T_n} \), and \( H \subset H_{T_n} \) for \( n \in \mathbb{N} \).
Now fix $n$. By hypothesis, for each pair $x, y \in C$ there exist $I, J \in H(\subset H_{Tn})$ and $u, v \in \{x, y\}$ such that

$$\|Tx - Ty\| \leq ||Iu - Jv||,$$

so

$$\|T_n x - T_n y\| \leq k_n ||Iu - Jv||,$$

by (4.11). Therefore, since $T_n$ is continuous and $k_n \in (0, 1)$, Corollary 3.9 applied to $T_n$ and $H_{Tn}$ (C compact implies that $C$ is bounded and complete) implies that there exists a unique $x_n \in C$ such that

$$x_n = T_n (x_n) = I(x_n) \quad \forall I \in H_{Tn}.$$  \hspace{1cm} (4.15)

Thus we have a sequence $\{x_n\}$ in $C$ which satisfies (4.15). Since $C$ is compact, $\{x_n\}$ has a subsequence $\{x_{i_n}\}$ which converges to some $a \in C$. Equations (4.11) and (4.15) thus imply that

$$a = \lim_{n \to \infty} x_{i_n} = \lim_{n \to \infty} k_{i_n} Tx_{i_n} + \lim_{n \to \infty} (1 - k_{i_n}) q = \lim_{n \to \infty} Ix_{i_n}. \hspace{1cm} (4.16)$$

But $T$ is continuous, so (4.16) implies that $a = Ta$, and $a = Ia$ for all continuous $I$. \hfill \Box

**Remark 4.8.** We see that Theorem 4.7 does indeed extend Theorem 3 in [7] if we observe that the family $\mathcal{F}$ in Theorem 3 [7]. is a family of sets which is a subset of $C_\phi$. We can let

$$H = \{\text{maps } h : C \to C \mid h \text{ is affine, } h \in C_\phi\}. \hspace{1cm} (4.17)$$

Then $H$ is a semi-group and $\mathcal{F} \subset H$.

**5. Conclusion.** We conclude with further evidence of the generality and applicability of the concept of being nearly commutative at a function $\varphi$. The theorem below generalizes Theorem 4.2 in [5] by replacing the semi-group $C_\varphi$ with a more general semi-group $H$.

**Theorem 5.1.** Let $f$ and $g$ be commuting self maps of a compact metric space $(X, d)$ such that $gf$ is continuous. If $H$ is a semi-group of self maps of $X$ which is n.c. at $gf$, and

$$fx \neq gy \Rightarrow d(fx, gy) < \delta(H(x, y)), \hspace{1cm} (5.1)$$

then there exists a unique point $a \in X$ such that $a = fa = ga = ha$ for all $h \in H$.

We leave the proof of Theorem 5.1 to the interested reader.

**References**


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