ON THE STRUCTURE OF THE TOTALLY ORDERED SET OF UNIMODAL CYCLES

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ABSTRACT. We continue the study of a class of unimodal cycles where each cycle in the class is forced by every unimodal cycle not in the class. For every order, we identify the cycle in the class of that order, which is maximal with respect to the forcing relation.

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1. Introduction. In 1964, Šarkovs'kiĭ defined a linear order on the set of natural numbers:

$$3 \prec 5 \prec 7 \prec \cdots \prec 2 \cdot 3 \prec 2 \cdot 5 \cdots \prec \cdots \prec 2^2 \cdot 3 \prec 2^2 \cdot 5 \cdots \prec \cdots \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1 \tag{1.1}$$

and proved the following theorem.

THEOREM 1.1 (Šarkovs'kiĭ [5]). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. The set of (least) periods of f is a tail of the above order. Conversely, for every tail of the above order, there is a continuous map $f : \mathbb{R} \to \mathbb{R}$ having exactly those periods.*

In 1987, Baldwin [2] considered not only the least period of a periodic point, but also the orbit type. He defined the forcing relation on finite cyclic permutations (cycles), proved that the forcing relation is a partial order, and provided an algorithm to decide when one cycle forces another.

A cycle is *unimodal* if the canonical linear map it determines has exactly one turning point. Throughout this paper, we assume a unimodal cycle has exactly one turning point and it is a maximum. It is shown in [3] that the forcing relation is a total order on the set of unimodal cycles.

In [4], we described a class of unimodal cycles, where every cycle in the class is forced by every unimodal cycle not in the class. In this paper, for each order n, we identify the maximal cycle of order n in this class with respect to the forcing relation.

2. Definitions. Let $f : I \to I$ be a continuous map of a compact interval to itself. We define $f^0(x) = x$ and for $n \in \mathbb{N}$, $n \geq 1$, $f^n(x) = f(f^{n-1}(x))$. If there exists $k \in \mathbb{N}$ such that $f^k(x) = x$, then we say x is periodic for f and x has least period s, where s is smallest element of $\mathbb{N}$ such that $f^s(x) = x$. For $x \in I$, the orbit of x is the set $\{f^n(x) \mid n \geq 0\}$. If x is periodic with least period s, then the orbit of x is the finite set $X = \{x, f(x), f^2(x), \ldots, f^{s-1}(x)\}$.

A cycle of order m is a bijection $\eta : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$ such that $\eta^k(1) \neq 1$ for $1 \leq k < m$. We denote a cycle η by $\eta = (k_1, k_2, \ldots, k_m)$, where $\eta(k_i) = k_{i+1}$ and

$\eta(k_m) = k_1$. We assume, without loss of generality, that $k_1 = 1$. Write the elements of a periodic orbit X in increasing order: $x_1 < x_2 < \cdots < x_s$. We say X has orbit type η if η is a cycle of order s and for each $i \in \{1,2,\ldots,s\}$, $f(x_i) = x_{\eta(i)}$. In fact, we say each $x_i \in X$ has orbit type η.

The forcing relation on cycles is defined as follows: θ forces η if and only if every continuous map of the interval that has a periodic orbit of type θ has a periodic orbit of type η. Baldwin [2] proved that this relation induces a partial order on the set of cycles.

Let θ be a cycle of order n. The canonical θ-linear map L_θ is defined by $L_\theta : [1,n] \to [1,n]$, where $L_\theta = \theta$ on $\{1,2,\ldots,n\}$ and L_θ is linear on $[i, i+1]$ for $1 \le i \le n-1$. It is known that θ forces η if and only if L_θ has a periodic orbit of orbit type η [1]. A cycle θ is called unimodal if L_θ has exactly one turning point. Throughout this paper, every unimodal cycle is assumed to have one turning point and it is a maximum. The forcing relation defined by Baldwin induces a total order on the set of unimodal cycles [3].

Let θ be a cycle of order n.

Definition 2.1. The RL-pattern for θ is the element $S = S_1 S_2 \cdots S_n \in \{R,L\}^n$ satisfying

$$S_i = \begin{cases} R & \text{if } \theta^i(1) > \theta^{i-1}(1), \\ L & \text{if } \theta^i(1) < \theta^{i-1}(1). \end{cases} \qquad (2.1)$$

For example, the RL-pattern for $\theta = (12354)$ is $RRRLL$. Both $\theta = (15234)$ and $\theta = (13245)$ have the RL-pattern $RLRRL$. Every RL-pattern begins with R and ends with L. If X is a periodic orbit of orbit type θ, then the RL-pattern for X is the RL-pattern for θ.

3. Preliminaries. The next four results are proved in [4].

Lemma 3.1. *If θ is unimodal, then L_θ has a unique fixed point.*

Lemma 3.2. *If θ is unimodal, then the RL-pattern for θ cannot contain two consecutive L's.*

Definition 3.3. Let C denote the class of unimodal cycles whose RL-pattern does not contain two consecutive R's.

Lemma 3.4. *Let $\theta \in C$ be of order n. Then*
(1) *n is even and $n \ge 4$.*
(2) *$\theta(n-i) = i+1$ for $i = 0,1,2,\ldots,k-1$.*
(3) *$\theta(1) = k+1$.*

Theorem 3.5. *Let $\eta \in C$ and let θ be a unimodal cycle not in C. Then θ forces η.*

Definition 3.6. For even $n \ge 4$,

$$\bar{\theta}_n = (1, k+1, k, k+2, k-1, k+3, k-2, \ldots, n-1, 2, n). \qquad (3.1)$$

Remark 3.7. In the main theorem, we prove that $\bar{\theta}_n$ is the maximal cycle of order n in C with respect to the forcing relation.

LEMMA 3.8. *Let $\theta \in C$ be of order n. Then $\theta = \bar{\theta}_n$ if and only if $\theta(2) = n$.*

PROOF. The map L_θ is a bijection which maps $\{3, 4, \ldots, k\}$ to $\{k+2, k+3, \ldots, n-1\}$. Since $\theta(2) = n$ and θ is unimodal, L_θ is decreasing on $\{3, 4, \ldots, k\}$. $\qquad\square$

The next lemma restates Lemma 3.8 in the way we will often use it.

LEMMA 3.9. *Let $\theta \in C$ be of order n. Then $\theta \neq \bar{\theta}_n$ if and only if there exists $i \in \{3, 4, \ldots, k\}$ such that $\theta(i) = n$.*

4. Main theorem

LEMMA 4.1. *Let $\theta \in C$ be of order n. Then the only periodic point for L_θ in $(k, k+1)$ is the unique fixed point for L_θ.*

PROOF. $L_\theta(k+1) = k$ and $L_\theta(k) \geq k+2$. If $x < p$, then for $j \geq 0$, $L_\theta^{2j}(x) < p$ and $L_\theta^{2j+1}(x) > p$. Let $\bar{k}$ be the unique preimage of $k+1$ in (k, p). If $x \in (k, \bar{k}]$, then $L_\theta(x) \geq k+1$ and x is not periodic since $[1, k] \cup [k+1, n]$ is invariant under L_θ. If $x \in (\bar{k}, p)$, then $L_\theta^2(x) < p$. If $L_\theta^2(x) < \bar{k}$, then x is not periodic. If $L_\theta^2(x) \in (\bar{k}, p)$, then $(p - L_\theta^2(x)) \geq 4(p - x)$ since $L_\theta' \leq -2$ on $(k, k+1)$. $\qquad\square$

REMARK 4.2. In the next few results, we assume $\theta \in C$ is of order n, $\theta \neq \bar{\theta}_n$, and θ forces $\bar{\theta}_n$. By Definition 3.6 this means that L_θ has a periodic orbit X of type $\bar{\theta}_n$. As standard notation, we write the elements of X in increasing order and label them as follows:

$$X = \{x_1 < x_2 < \cdots < x_n\}, \tag{4.1}$$

where

$$L_\theta(x_i) = x_{\bar{\theta}_n(i)}. \tag{4.2}$$

LEMMA 4.3. *Suppose that $\theta \in C$ is of order n and $\theta \neq \bar{\theta}_n$. If θ forces $\bar{\theta}_n$, then $\{x_1, x_2, \ldots, x_k\} \subseteq (1, k)$ and $\{x_{k+1}, x_{k+2}, \ldots, x_n\} \subseteq (k+1, n)$.*

PROOF. Since $X = \{x_1 < x_2 < \cdots < x_n\}$ is an orbit of type $\bar{\theta}_n$, we have

$$\{x_1, x_2, \ldots, x_k\} \subseteq \{x \mid L_\theta(x) > x \text{ and } L_\theta^2(x) < L_\theta(x)\} \subseteq [1, p],$$
$$\{x_{k+1}, x_{k+2}, \ldots, x_n\} \subseteq \{x \mid L_\theta(x) < x \text{ and } L_\theta^2(x) > L_\theta(x)\} \subseteq [p, n]. \tag{4.3}$$

But by Lemma 4.1, no x_i can be in $(k, k+1)$. $\qquad\square$

LEMMA 4.4. *Let $\theta \in C$ be of order n and $\theta \neq \bar{\theta}_n$. If θ forces $\bar{\theta}_n$, then there are at most two x_i in $(1, 3)$.*

PROOF. If not, then (at least) $x_1 < x_2 < x_3$ are in $(1, 3)$. By Lemma 3.9, L_θ is increasing on $(1, 3)$. So $L_\theta(x_3) > L_\theta(x_2)$ or $x_{\bar{\theta}_n(3)} > x_{\bar{\theta}_n(2)}$ or $x_{n-1} > x_n$. Hence, there are at most two x_i in $(1, 3)$. $\qquad\square$

The next definition and lemma could have been proved just after Lemma 3.4, but we place them here since they were not needed until now.

DEFINITION 4.5. For $\theta \in C$ of order n, let $P_j = \{i \mid \theta(i) \geq j\}$.

LEMMA 4.6. *Let $\theta \in C$ be of order n. Then*
(1) *the order of P_j is $n - (j-1)$,*
(2) *for $1 \leq j \leq n$, either the largest element in P_j or the smallest element in P_j maps to j under L_θ.*

PROOF. The proof of (1) is obvious.

(2) is proved inductively. We know that $P_1 = \{1, 2, \ldots, n\}$ and $L_\theta(n) = 1$, so (2) is true for $j = 1$. For $j > 1$, P_j is a finite sequence of consecutive integers since each P_j is derived from P_{j-1} by removing the maximum or minimum of P_{j-1} from P_{j-1}. Assume that (2) is true for j. Consider $P_{j+1} = \{i \mid \theta(i) \geq j+1\}$. Let $a = \min P_{j+1}$ and $b = \max P_{j+1}$. If neither a nor b maps to $j+1$ under L_θ, then $L_\theta(a) > j+1$ and $L_\theta(b) > j+1$. But there exists an integer $c \in (a,b)$ such that $L_\theta(c) = j+1$, and this contradicts the fact that θ is unimodal. $\square$

LEMMA 4.7. *Let $\theta \in C$ be of order n and let $k = n/2$. Let $\bar{j} \geq 0$ be the largest integer such that $2 + \bar{j} < k - \bar{j}$. Then for all $j = 0, 1, 2, \ldots, \bar{j}$,*

$$L_\theta((2+j, k-j)) \subseteq (k+2+j, n]. \tag{4.4}$$

PROOF. For $j = 0$, consider, by Lemma 3.4,

$$P_{k+2} = \{i \mid \theta(i) \geq k+2\} = \{2, 3, \ldots, k\}. \tag{4.5}$$

So if $x \in (2, k)$, then $L_\theta(x) > k+2$. That is,

$$L_\theta((2, k)) \subseteq (k+2, n]. \tag{4.6}$$

Note that P_{k+3} properly contains $\{3, 4, \ldots, k-1\}$ since by Lemma 4.6 exactly one of $\{2, k\}$ maps to $k+2$ under L_θ.

Inductively, suppose that j satisfies $0 \leq j < \bar{j}$ and

$$L_\theta((2+j, k-j)) \subseteq (k+2+j, n]. \tag{4.7}$$

Consider

$$P_{k+2+(j+1)} = \{i \mid \theta(i) \geq k+2+(j+1)\} \supset \{2+(j+1), \ldots, k-(j+1)\} \tag{4.8}$$

(a proper containment). So if $x \in (2+(j+1), k-(j+1))$, then

$$L_\theta(x) > k+2+(j+1). \tag{4.9}$$

That is,

$$L_\theta((2+(j+1), k-(j+1))) \subseteq (k+2+(j+1), n], \tag{4.10}$$

as needed. $\square$

LEMMA 4.8. *Let $\theta \in C$ be of order n, and let $k = n/2$. Assume that $\theta \neq \bar{\theta}_n$ and θ forces $\bar{\theta}_n$. Let $\bar{j} \geq 0$ be the largest integer such that $2 + \bar{j} < k - \bar{j}$. Then for all $j = 0, 1, 2, \ldots, \bar{j}$, there exists at most one x_i in $(k-j-1, k-j)$.*

PROOF. We have $\theta \in C$ of order n, $\theta \neq \bar{\theta}_n$, and θ forces $\bar{\theta}_n$. By Lemma 4.3, $\{x_1, x_2, \ldots, x_k\} \subseteq (1, k)$. Let $\bar{j}$ be the largest integer such that $2 + j < k - \bar{j}$. The proof is inductive. First, for $j = 0$, suppose that there are two or more $x_i \in (k-1, k)$. Then (at least) x_k, x_{k-1} are in $(k-1, k)$. It follows from Lemma 3.4 that

$$x_{i+1}, x_{k+2} \in (k+1, k+2) \tag{4.11}$$

since x_{k+1} and x_{k+2} are preimages of x_k and x_{k-1} and

$$L_\theta^{-1}([k-1, k]) = [k+1, k+2]. \tag{4.12}$$

Now, we have $x_k \in (k-1, k)$ and

$$\theta(x_k) = x_{k+2} \in (k+1, k+2). \tag{4.13}$$

But

$$\theta(x_k) \in L_\theta(k-1, k) \subseteq L_\theta(2, k) \subseteq (k+2, n] \tag{4.14}$$

by Lemma 4.7, which is a contradiction. Therefore, there is at most one $x_i \in (k-1, k)$.

Next, suppose that $\tilde{j}$ satisfies $0 \leq \tilde{j} < \bar{j}$ and for all $j = 0, 1, 2, \ldots, \tilde{j}$, there is at most one x_i in $(k-j-1, k-j)$. Suppose that there are two or more x_i in $J = (k-\tilde{j}-2, k-\tilde{j}-1)$. There are $\tilde{j}+2$ ways for this to occur, since each of the $\tilde{j}+1$ intervals

$$(k-1, k), (k-2, k-1), \ldots, (k-\tilde{j}-1, k-\tilde{j}) \tag{4.15}$$

contains at most one x_i. As in the case when $j = 0$, the preimages of any elements in $J = (k-\tilde{j}-2, k-\tilde{j}-1)$ must be in $K = (k+\tilde{j}+2, k+\tilde{j}+3)$.

CASE 1 $(x_k, x_{k-1} \in J)$. Then $x_{k+1}, x_{k+2} \in K$.

CASE 2 $(x_{k-1}, x_{k-2} \in J)$. Then $x_{k+2}, x_{k+3} \in K$.

and so on, until

CASE $\tilde{j}+2$ $(x_{k-\tilde{j}-1}, x_{k-\tilde{j}-2} \in J)$. Then $x_{k+\tilde{j}+2}, x_{k+\tilde{j}+3} \in K$.

In every case there exists $x_r \in J$ and $L_\theta(x_r) \in K$. Now,

$$L_\theta(x_r) \in L_\theta(J) = L_\theta(k-\tilde{j}-2, k-\tilde{j}-1) \subseteq L_\theta(2+\tilde{j}+1, k-\tilde{j}-1) \subseteq (k+2+\tilde{j}+1, n], \quad (4.16)$$

by Lemma 4.7. So $L_\theta(x_r) > k + \tilde{j} + 3$. But

$$L_\theta(x_r) \in K = (k+\tilde{j}+2, k+\tilde{j}+3), \tag{4.17}$$

which is a contradiction. This proves the lemma. Note that $2 + (\tilde{j}+1) < k - (\tilde{j}+1)$ by hypothesis, so $2 + \tilde{j} + 1 \leq k - \tilde{j} - 2$, as needed in (4.16). $\square$

LEMMA 4.9. *Let $\theta \in C$ be of order n, and let $k = n/2$. Assume that $\theta \neq \bar{\theta}_n$ and θ forces $\bar{\theta}_n$. Let $\bar{j}$ be the largest integer such that $2 + \bar{j} < k - \bar{j}$. If $\theta(i) = n$, then $3 \leq i \leq k - \bar{j} - 2$.*

PROOF. There are at most two x_i that are less than i, as in the proof of Lemma 4.4. So there are at least $(k-2)$ x_i in (i, k). If $i > k - \bar{j} - 2$, then there are at most $\bar{j} + 1$ intervals and each interval contains at most one x_i. Since $k - 2 > \bar{j} + 1$, this is a contradiction. $\square$

LEMMA 4.10. *Let $\theta \in C$ be of order n and let $k = n/2$. Assume that $\theta \neq \bar{\theta}_n$ and θ forces $\bar{\theta}_n$. Let $\bar{j}$ be the largest integer such that $2 + \bar{j} < k - \bar{j}$. Then there is at most one x_i in each of the intervals*

$$(3,4),(4,5),\dots,(k-\bar{j}-2,k-\bar{j}-1). \tag{4.18}$$

PROOF. Let $\bar{i}$ be such that $\theta(\bar{i}) = n$. Then $x_1 < \bar{i} < x_3$. By Lemma 4.9, $3 \leq \bar{i} \leq k - \bar{j} - 2$. Define $\tilde{i}$ by $\tilde{i} = k - \bar{j} - \bar{i}$. Suppose that there are at least two $x_i \in (\tilde{i}, \tilde{i} + 1)$.

Either

$$x_2, x_3 \in (k - (\bar{j} + \tilde{i}), k - (\bar{j} + \tilde{i}) + 1), \tag{4.19}$$

in which case

$$x_{n-1}, x_{n-2} \in (k + (\bar{j} + \tilde{i}), k + (\bar{j} + \tilde{i}) + 1), \tag{4.20}$$

or

$$x_3, x_4 \in (k - (\bar{j} + \tilde{i}), k - (\bar{j} + \tilde{i}) + 1), \tag{4.21}$$

in which case

$$x_{n-2}, x_{n-3} \in (k + (\bar{j} + \tilde{i}), k + (\bar{j} + \tilde{i}) + 1). \tag{4.22}$$

In either case, there exists $x_r \in (k - (\bar{j} + \tilde{i}), k - (\bar{j} + \tilde{i}) + 1)$ such that

$$\theta(x_r) \in (k + (\bar{j} + \tilde{i}), k + (\bar{j} + \tilde{i}) + 1). \tag{4.23}$$

This guarantees that

$$\theta(k - (\bar{j} + \tilde{i}) + 1) \leq k + \bar{j} + \tilde{i}. \tag{4.24}$$

But, by (2) in Lemma 3.4,

$$\theta(k + (\bar{j} + \tilde{i})) = \theta(n - k + (\bar{j} + \tilde{i})) = \theta(n - (k - \bar{j} - \tilde{i})) = k - \bar{j} - \tilde{i} + 1 = k - (\bar{j} + \tilde{i}) + 1. \tag{4.25}$$

Since $n \geq 4$,

$$\theta(k - (\bar{j} + \tilde{i}) + 1) < k + \bar{j} + \tilde{i}. \tag{4.26}$$

This leaves the set $A = \{2, 3, \dots, \bar{i} - 1\}$ to be mapped bijectively by θ onto a set $B \supseteq \{k + \bar{j} + \tilde{i}, \dots, n - 1\}$. But

$$|A| = \bar{i} - 2 = k - \bar{j} - \tilde{i} - 2,$$
$$|B| \geq (n-1) - (k + \bar{j} + \tilde{i}) + 1 = (n - k) - \bar{j} - \tilde{i} = k - \bar{j} - \tilde{i}. \tag{4.27}$$

Hence, there is at most one $x_i \in (\tilde{i}, \tilde{i} + 1)$.

The proof continues in this fashion. We have proved that there is at most one $x_i \in (\tilde{i}, \tilde{i} + 1)$. Suppose that there are at least two $x_i \in (\tilde{i} + 1, \tilde{i} + 2)$. There are three ways for this to occur. In each of these three cases, there exists $x_r \in (\tilde{i} + 1, \tilde{i} + 2)$ such that

$$\theta(x_r) \in (k + \bar{j} + \tilde{i} - 1, k + \bar{j} + \tilde{i}). \tag{4.28}$$

As in the previous case,

$$\theta(\tilde{i} + 2) \leq k + \bar{j} + \tilde{i} - 1. \tag{4.29}$$

This leaves the set $A = \{2, 3, \dots, \bar{i} - 1\} \cup \{\tilde{i} + 1\}$ with $|A| = \bar{i} - 1 = k - \bar{j} - \tilde{i} - 1$ to be mapped bijectively onto a set containing $B \supseteq \{k + \bar{j} + \tilde{i} - 1, \dots, n - 1\}$, where $|B| \geq k - \bar{j} + \tilde{i}$. $\square$

THEOREM 4.11. *Let $\theta \in C$ of order n. Then $\bar{\theta}_n$ forces θ.*

PROOF. If $\theta = \bar{\theta}_n$, the conclusion holds trivially. Assume that $\theta \neq \bar{\theta}_n$ and suppose that $\bar{\theta}_n$ does not force θ. Since forcing induces a total order on C, θ forces $\bar{\theta}_n$, so there exist $x_1 < x_2 < \cdots < x_n$ in $(1, n)$ such that for each i, $L_\theta(x_i) = x_{\bar{\theta}_n(i)}$. By Lemma 4.3, $\{x_1, x_2, \ldots, x_k\} \subseteq (1, k)$, where $k = n/2$. By Lemma 4.4, there are at most two $x_i \in (1, 3)$. This leaves at least $k - 2$ of the x_i in the union of intervals $(3, 4) \cup (4, 5) \cup \cdots \cup (k - 1, k)$. But each one of these $k - 3$ intervals contains at most one x_i by Lemmas 4.8 and 4.10.

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