ISHIKAWA ITERATION PROCESS WITH ERRORS FOR NONEXPANSIVE MAPPINGS

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ABSTRACT. We study the construction and the convergence of the Ishikawa iterative process with errors for nonexpansive mappings in uniformly convex Banach spaces. Some recent corresponding results are generalized.

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1. Introduction. Let C be a closed convex subset of a Banach space X and $T : C \rightarrow C$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all x, y in C). Recently, Deng and Li [1] introduced an Ishikawa iteration sequence with errors as follows: for any given $x_0 \in C$

$$\begin{aligned}x_{n+1} &= \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \\y_n &= \hat{\alpha}_n x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n v_n, \quad n \geq 0.\end{aligned}\tag{1.1}$$

Here $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in C , and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha}_n\}$, $\{\hat{\beta}_n\}$, and $\{\hat{\gamma}_n\}$ are six sequences in $[0, 1]$ satisfying the conditions

$$\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1 \quad \forall n \geq 0.\tag{1.2}$$

REMARK 1.1. Note that the Ishikawa iteration processes [2] is a special case of the Ishikawa iteration processes with errors.

Deng and Li [1] obtained the following result. Let C be a closed convex subset of a uniformly convex Banach space X . If for any initial guess $x_0 \in C$, $\{x_n\}$ defined by (1.1), with the restrictions that $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$, $\sum_{n=0}^{\infty} \alpha_n \beta_n \hat{\beta}_n < \infty$, $\sum_{n=0}^{\infty} \gamma_n < \infty$, and $\sum_{n=0}^{\infty} \hat{\gamma}_n < \infty$, then $\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0$. So Deng and Li extended the result of Tan and Xu [6].

In this paper, we first extend and unify [1, Theorem 1] and [6, Lemma 3]. Then, we generalize [1, Theorems 2, 3, and 4] and [6, Theorems 1, 2, and 3].

2. Lemmas

LEMMA 2.1 (see [6]). *Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n \rightarrow \infty} a_n$ exists.*

LEMMA 2.2 (see [1]). *Let C be a closed convex subset of a Banach space X , $T : C \rightarrow C$ a nonexpansive mapping. Then for any initial guess x_0 in C , $\{x_n\}$ defined by (1.1),*

$$\|x_{n+1} - p\| \leq \|x_n - p\| + \gamma_n \|u_n - p\| + \beta_n \hat{\gamma}_n \|v_n - p\| \quad (2.1)$$

for all $n \geq 1$ and for all $p \in F(T)$, where $F(T)$ denotes the set of fixed point of T .

REMARK 2.3. Since the sequences $\{u_n\}$ and $\{v_n\}$ are bounded, so the sequences $\{\|u_n - p\|\}$ and $\{\|v_n - p\|\}$ are bounded too, then $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists by Lemma 2.1.

LEMMA 2.4 (see [7]). *Let C be a bounded closed convex subset of a uniformly convex Banach space X . Suppose that $T : C \rightarrow C$ is a nonexpansive mapping. If $\gamma_n \rightarrow \gamma$ weakly ($\gamma_n, \gamma \in C$, $n = 1, 2, \dots$), then there exists a strictly increasing convex function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $g(0) = 0$ such that*

$$g(\|\gamma - T\gamma\|) \leq \liminf_{n \rightarrow \infty} \|\gamma_n - T\gamma_n\|. \quad (2.2)$$

3. Main results

THEOREM 3.1. *Let C be a closed convex subset of a uniformly convex Banach space X , $T : C \rightarrow C$ a nonexpansive mapping with a fixed point. If for any initial guess x_0 in C , $\{x_n\}$ defined by (1.1), with the restrictions that $\sum_{n=0}^{\infty} \gamma_n < \infty$, $\sum_{n=0}^{\infty} \hat{\gamma}_n < \infty$, and there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\sum_{k=0}^{\infty} \alpha_{n_k} \beta_{n_k} = \infty$, $\sum_{k=0}^{\infty} \alpha_{n_k} \beta_{n_k} \hat{\beta}_{n_k} < \infty$. Then $\liminf_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$.*

PROOF. Since T has a fixed point, and by Lemma 2.2, we may set

$$M = \sup_{n \geq 0} \{\|Tx_n - u_n\|, \|x_n - u_n\|, \|Ty_n - v_n\|, \|\gamma_n - u_n\|, \|x_n - v_n\|\}. \quad (3.1)$$

If $\liminf_{n \rightarrow \infty} \|x_n - Tx_n\| > 0$, we may assume that $\liminf_{n \rightarrow \infty} \|x_n - p\| > 0$, where $p \in F(T)$. Since $\|Ty_n - p\| \leq \|x_n - p\| + \hat{\gamma}_n M$, we obtain

$$\begin{aligned} \|x_{n+1} - p\| &\leq \|\alpha_n(x_n - p) + \beta_n(Ty_n - p)\| + \gamma_n M \\ &= (\alpha_n + \beta_n) \left\| \frac{\alpha_n}{\alpha_n + \beta_n} (x_n - p) + \frac{\beta_n}{\alpha_n + \beta_n} (Ty_n - p) \right\| + \gamma_n M \\ &\leq \left[1 - 2 \frac{\alpha_n \beta_n}{(\alpha_n + \beta_n)^2} \delta_X \left(\frac{\|x_n - Ty_n\|}{\|x_n - p\| + \hat{\gamma}_n M} \right) \right] (\|x_n - p\| + \hat{\gamma}_n M) + \gamma_n M \quad (3.2) \\ &\leq \left[1 - 2 \alpha_n \beta_n \delta_X \left(\frac{\|x_n - Ty_n\|}{\|x_n - p\| + \hat{\gamma}_n M} \right) \right] \|x_n - p\| + (\hat{\gamma}_n + \gamma_n) M, \end{aligned}$$

where δ_X is the modulus of convexity of the uniformly convex Banach space X . Setting

$$D_n = 1 - 2 \alpha_n \beta_n \delta_X \left(\frac{\|x_n - Ty_n\|}{\|x_n - p\| + \hat{\gamma}_n M} \right). \quad (3.3)$$

Thus for all $n \geq 0$, $0 \leq D_n \leq 1$. From (3.2), for all $k \geq 0$, we have

$$\begin{aligned}
 & \|x_{n_{k+1}} - p\| \\
 & \leq D_{n_{k+1}-1} \|x_{n_{k+1}-1} - p\| + (\hat{y}_{n_{k+1}-1} + y_{n_{k+1}-1})M \\
 & \leq D_{n_{k+1}-1} D_{n_{k+1}-2} \cdots D_{n_k+1} D_{n_k} \|x_{n_k} - p\| + \sum_{i=1}^{n_{k+1}-n_k} (\hat{y}_{n_{k+1}-i} + y_{n_{k+1}-i})M \\
 & \leq D_{n_k} \|x_{n_k} - p\| + \sum_{i=1}^{n_{k+1}-n_k} (\hat{y}_{n_{k+1}-i} + y_{n_{k+1}-i})M \\
 & \leq \|x_{n_k} - p\| \left[1 - 2\alpha_{n_k} \beta_{n_k} \delta_X \left(\frac{\|x_{n_k} - Ty_{n_k}\|}{\|x_{n_k} - p\| + \hat{y}_{n_k} M} \right) \right] + \sum_{i=1}^{n_{k+1}-n_k} (\hat{y}_{n_{k+1}-i} + y_{n_{k+1}-i})M.
 \end{aligned} \tag{3.4}$$

Thus,

$$\begin{aligned}
 & \sum_{i=0}^k 2\alpha_{n_i} \beta_{n_i} \delta_X \left(\frac{\|x_{n_i} - Ty_{n_i}\|}{\|x_{n_i} - p\| + \hat{y}_{n_i} M} \right) \|x_{n_i} - p\| \\
 & \leq \|x_{n_0} - p\| - \|x_{n_{k+1}} - p\| + \sum_{i=0}^{n_{k+1}-1} (\hat{y}_i + y_i)M.
 \end{aligned} \tag{3.5}$$

It follows that

$$\sum_{i=0}^{\infty} \alpha_{n_i} \beta_{n_i} \delta_X \left(\frac{\|x_{n_i} - Ty_{n_i}\|}{\|x_{n_i} - p\| + \hat{y}_{n_i} M} \right) < +\infty. \tag{3.6}$$

By condition $\sum_{i=0}^{\infty} \alpha_{n_i} \beta_{n_i} \hat{\beta}_{n_i} < +\infty$, we have

$$\sum_{i=0}^{\infty} \alpha_{n_i} \beta_{n_i} \left[\delta_X \left(\frac{\|x_{n_i} - Ty_{n_i}\|}{\|x_{n_i} - p\| + \hat{y}_{n_i} M} \right) + \hat{\beta}_{n_i} \right] < +\infty. \tag{3.7}$$

It follows that

$$\liminf_{k \rightarrow \infty} \left[\delta_X \left(\frac{\|x_{n_k} - Ty_{n_k}\|}{\|x_{n_k} - p\| + \hat{y}_{n_k} M} \right) + \hat{\beta}_{n_k} \right] = 0 \tag{3.8}$$

since $\sum_{k=0}^{\infty} \alpha_{n_k} \beta_{n_k} = \infty$. Hence, there is a sequence $\{n_{k_i}\} \subset \{n_k\}$ such that

$$\lim_{i \rightarrow \infty} \|x_{n_{k_i}} - Ty_{n_{k_i}}\| = 0, \quad \lim_{i \rightarrow \infty} \hat{\beta}_{n_{k_i}} = 0. \tag{3.9}$$

On the other hand, we have

$$\begin{aligned}
 \|x_{n_{k_i}} - Tx_{n_{k_i}}\| & \leq \|x_{n_{k_i}} - Ty_{n_{k_i}}\| + \|Tx_{n_{k_i}} - Ty_{n_{k_i}}\| \\
 & \leq \|x_{n_{k_i}} - Ty_{n_{k_i}}\| + \hat{\beta}_{n_{k_i}} \|x_{n_{k_i}} - Tx_{n_{k_i}}\| + \hat{y}_{n_{k_i}} M.
 \end{aligned} \tag{3.10}$$

Setting $i \rightarrow \infty$ in (3.10), it follows from (3.9) that

$$\lim_{i \rightarrow \infty} \|x_{n_{k_i}} - Tx_{n_{k_i}}\| = 0. \tag{3.11}$$

Thus,

$$\liminf_{n \rightarrow \infty} \|x_n - Tx_n\| = 0. \quad (3.12)$$

This completes the proof. $\square$

Recall that a Banach space X is said to satisfy Opial's condition [4] if the condition $x_n \rightharpoonup x_0$ weakly implies

$$\limsup_{n \rightarrow \infty} \|x_n - x_0\| < \limsup_{n \rightarrow \infty} \|x_n - y\| \quad \forall y \neq x_0. \quad (3.13)$$

A mapping $T : C \rightarrow C$ with a nonempty fixed points set $F(T)$ in C will be said to satisfy Condition A in [5] if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$, such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf\{\|x - z\| : z \in F(T)\}$.

THEOREM 3.2. *Let C be a bounded closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition or whose norm is Fréchet differentiable. Let $T : C \rightarrow C$ a nonexpansive mapping with a fixed point, and $\{x_n\}$ defined by (1.1), with the restrictions that $\sum_{n=0}^{\infty} \gamma_n < \infty$, $\sum_{n=0}^{\infty} \hat{\gamma}_n < \infty$, and for any subsequence $\{n_k\}$ of $\{n\}$, $\sum_{k=0}^{\infty} \alpha_{n_k} \beta_{n_k} = \infty$, $\sum_{k=0}^{\infty} \alpha_{n_k} \beta_{n_k} \hat{\beta}_{n_k} < \infty$, converges weakly to a fixed point of T .*

By Theorem 3.1 and Lemma 2.4, we can prove Theorem 3.2 easily. The proof is similar to that of [7, Theorem 3.1], so the details are omitted.

Let X, C, T , and $\{x_n\}$ be as in Theorem 3.1. Then we have the following theorem.

THEOREM 3.3. *If the range of C under T is contained in a compact subset of X , then $\{x_n\}$ converges strongly to a fixed point of T .*

THEOREM 3.4. *Let C be a bounded closed convex subset of a uniformly convex Banach space X . If T satisfies Condition A, then $\{x_n\}$ converges strongly to a fixed point of T .*

PROOF. Since C is a bounded closed convex subset of a uniformly convex Banach space X , then T has a fixed point [3]. So $F(T)$ is nonempty. It follows from Theorem 3.1 and Condition A, that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k \rightarrow \infty} f(d(x_{n_k}, F(T))) = 0$, therefore we have $\lim_{k \rightarrow \infty} d(x_{n_k}, F(T)) = 0$. So we can choose a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ and some sequence $\{p_i\}$ in $F(T)$ such that $\|x_{n_{k_i}} - p_i\| < 2^{-i}$ for all integers $k \geq 0$.

We denote $\sup_n \{\|u_n - p\|, \|v_n - p\|\}$ by M and $(\gamma_{n_{k_i}} + \beta_{n_{k_i}} \hat{\gamma}_{n_{k_i}})M$ by $\lambda_{n_{k_i}}$. By Lemma 2.1 we have

$$\begin{aligned} \|p_{i+1} - p_i\| &\leq \|x_{n_{k_{i+1}}} - p_{i+1}\| + \|x_{n_{k_{i+1}}} - p_i\| \\ &\leq 2^{-(i+1)} + \|x_{n_{k_{i+1}}-1} - p_i\| + \lambda_{n_{k_{i+1}}-1} \\ &\leq 2^{-(i+1)} + \|x_{n_{k_{i+1}}-2} - p_i\| + \lambda_{n_{k_{i+1}}-2} + \lambda_{n_{k_{i+1}}-1} \end{aligned}$$

$$\begin{aligned}
&\leq 2^{-(i+1)} + \left\| x_{n_{k_i}} - p_i \right\| + \sum_{j=n_{k_i}}^{n_{k_{i+1}}-1} \lambda_j \\
&\leq 2^{-i+1} + \sum_{j=n_{k_i}}^{n_{k_{i+1}}-1} \lambda_j.
\end{aligned}
\tag{3.14}$$

It follows, from (3.14) and $\sum_j \lambda_j$ is convergent, that $\{p_i\}$ is a Cauchy sequence therefore converges strongly to a point $p \in F(T)$, since $F(T)$ is closed. We have seen that $\{x_{n_{k_i}}\}$ converges strongly to p , so does $\{x_n\}$ by the Remark 2.3. This completes the proof. $\square$

REMARK 3.5. The above three theorems generalize [6, Theorems 1, 2, and 3] and [1, Theorems 2, 3, and 4], respectively.

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