BIHARMONIC MAPS ON V-MANIFOLDS

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ABSTRACT. We generalize biharmonic maps between Riemannian manifolds into the case of the domain being V-manifolds. We obtain the first and second variations of biharmonic maps on V-manifolds. Since a biharmonic map from a compact V-manifold into a Riemannian manifold of nonpositive curvature is harmonic, we construct a biharmonic non-harmonic map into a sphere. We also show that under certain condition the biharmonic property of f implies the harmonic property of f . We finally discuss the composition of biharmonic maps on V-manifolds.

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1. Introduction. Following Eells, Sampson, and Lemaire's tentative ideas [7, 8, 9], Jiang first discussed biharmonic (or 2-harmonic) maps between Riemannian manifolds in his two articles [10, 11] in China in 1986, which gives the conditions for biharmonic maps. A biharmonic map $f : M \rightarrow N$ between Riemannian manifolds is the smooth critical point of the bi-energy functional

$$E_2(f) = \int_M \|(d + d^*)f\|^2 * 1 = \int_M \|\tau(f)\|^2 * 1, \quad (1.1)$$

where $*1$ is the volume form on M , the tension field $\tau(f) = (\hat{D}df)(e_i, e_i) = (\hat{D}_{e_i}df)(e_i)$, $\{e_i\}$ is the local frame of a point p in M . Biharmonic maps are the extensions of harmonic maps, and their study provides a source in partial differential equations, differential geometry, and analysis. After Jiang, Chiang, and Sun have studied biharmonic maps in two papers [6, 14]. Chiang also studied harmonic maps and biharmonic maps of two different kinds of singular spaces: V-manifolds [3, 4] and spaces with conical singularities (with Andrea Ratto [5]).

In this paper, we generalize the notion of a biharmonic map to the case that the domain of f is a V-manifold due to Satake in [1, 12, 13]. A (C^∞) V-manifold $(M, \mathcal{F})$ consists of a Hausdorff space M with an atlas $\mathcal{F}$ of V-charts satisfying the following conditions:

- (i) If $\{\tilde{U}, G, \pi\}$ and $\{\tilde{U}', G', \pi'\}$ are two V-charts in $\mathcal{F}$ over U, U' , respectively, in M with $U \subset U'$, then there exists an injection $\lambda : \{U, G, \pi\} \rightarrow \{U', G', \pi'\}$.
- (ii) The supports of V-charts in $\mathcal{F}$ form a basis for open sets in M .

Take a chart $\{\tilde{U}, G, \pi\} \in \mathcal{F}$ such that $p \in \pi(\tilde{U})$ and choose $\tilde{p} \in \tilde{U}$ such that $\sigma\tilde{p} = \tilde{p}$. The isotropic subgroup $G_{\tilde{p}}$ of G at $\tilde{p}$ is the set of all $\sigma \in G$ such that $\sigma\tilde{p} = \tilde{p}$. So $G_{\tilde{p}}$ is called the *isotropic group* of p . The singular set $\mathbb{S}$ of M consists of all singular points of M , that is, the points of M with nontrivial isotropy groups. (For example, S^2/Z_3 is a compact V-manifold with two singular points.) The main difficulties of this

paper arise from the complicated behavior of the singular locus of V-manifolds, and therefore a different method than the usual one is required. In fact, this article is the extension of Chiang's previous two papers [3, 4].

We derive the first variations of biharmonic maps in [Theorem 2.2](#), and give the definition for biharmonic maps on V-manifolds. We show that a biharmonic map from a compact V-manifold into a Riemannian manifold of nonpositive curvature is a harmonic map in [Theorem 2.4](#). Then we construct a biharmonic non-harmonic map from a V-manifold into a sphere in [Section 2](#). We obtain the second variations of biharmonic maps in [Theorem 3.1](#). If $d^2/dt^2 E_2(f_t)|_{t=0} \geq 0$, then f is a stable biharmonic map. In [Theorem 3.3](#), we show that if a stable biharmonic map from a compact V-manifold M into a Riemannian manifold N of positive curvature satisfies the conservation law, then f must be a harmonic map. In [Theorem 3.4](#), we prove the composition of biharmonic maps on V-manifolds which generalizes Sun's result in [14].

2. Biharmonic maps on V-manifolds. Let $(M, \mathcal{F})$ be a $(\mathbb{C}^\infty)$ V-manifold, and U be an open subset of M . By a V-chart on M over U we mean a system $\{\tilde{U}, G, \pi\}$ consisting of (1) a connected open subset $\tilde{U}$ of $\mathbb{R}^m$, (2) a finite group G of diffeomorphisms of $\tilde{U}$, with the set of fixed points of codimension ≥ 2 , and (3) a continuous map of $\pi : \tilde{U} \rightarrow U$ such that $\pi \circ \sigma = \pi$ for $\sigma \in G$ and such that π induces a homeomorphism of $\tilde{U}/G$ onto U . The set U is called the *support* of V-chart, and π is called the *projection* onto U .

Let $(M, \mathcal{F})$ be a V-manifold and $p \in M$. Take a chart $\{\tilde{U}, G, \pi\} \in \mathcal{F}$ such that $p \in \pi(\tilde{U})$ and choose $\tilde{p} \in \tilde{U}$ such that $\pi(\tilde{p}) = p$. The isotropic subgroup $G_{\tilde{p}}$ of G at $\tilde{p}$ is the set of all $\sigma \in G$ such that $\sigma\tilde{p} = \tilde{p}$, and is uniquely determined by p . Therefore, $G_{\tilde{p}}$ is called the *isotropic group* of p . The singular set $\mathbb{S}$ of M consists of all singular points of M , that is, the points of M with nontrivial isotropic groups. Let $(\tilde{x}^1, \dots, \tilde{x}^m)$ be a coordinate system around $\tilde{p}$ and consider the system $\tilde{y}^i = 1/|G_{\tilde{p}}| \sum l_{ij}(\sigma^{-1})\tilde{x}^j \cdot \sigma$ with

$$l_{ij}(\sigma) = \left[\frac{\partial \tilde{x}^i \circ \sigma}{\partial \tilde{x}^j} \right]_{\tilde{p}}, \quad |G_{\tilde{p}}| = \text{order of } G_{\tilde{p}}. \quad (2.1)$$

Then the $\{\tilde{y}^i\}$ are a new coordinate system around $\tilde{p}$ and $G_{\tilde{p}}$ operates linearly in the $\tilde{y}$ -system. After this suitable C^∞ change of coordinates around $\tilde{p}$, $G_{\tilde{p}}$ becomes a finite group of linear transformations. The fixed point set of any $\sigma \in G_{\tilde{p}}$ is the defined linear equations in the $\tilde{y}$, and consequently the fixed point set of $\sigma \in G_{\tilde{p}}$ in $\tilde{U}$ is the intersection of $\tilde{U}$ with a linear space. Therefore, $\pi^{-1}\mathbb{S}$ is locally expressed by a finite union of linear spaces intersected with $\tilde{U}$. Hence $\mathbb{S}$ is a V-submanifold of codimension ≥ 2 of M . Clearly, $M - \mathbb{S}$ is an ordinary manifold.

We fix a V-manifold M with defining atlas $\mathcal{F}$. A *smooth function* $f : (M, \mathcal{F}) \rightarrow N$ from M into an ordinary manifold N is defined as follows: for any $\{\tilde{U}, G, \pi\} \in \mathcal{F}$ there corresponds an ordinary G -invariant smooth map $f_{\tilde{U}}^G = 1/|G| \sum_{\sigma \in G} f_{\tilde{U}} \circ \sigma : \tilde{U} \rightarrow N$ such that $f_{\tilde{U}}^G = f \circ \pi$ and $f_{\tilde{U}}^G = f_{\tilde{U}'}^{G'} \circ \lambda$ for any injection $\lambda : \{\tilde{U}, G, \pi\} \rightarrow \{\tilde{U}', G', \pi'\}$ where $f_{\tilde{U}} : \tilde{U} \rightarrow N$ is an ordinary smooth map.

Put a Riemannian metric $g_{\tilde{U}} = g_{ij} d\tilde{x}^i d\tilde{x}^j$ on $\tilde{U}$. By taking the G -average if necessary, we can assume that $g_{\tilde{U}}$ is G -invariant. Thus the transformations $\sigma \in G$ are isometries for $g_{\tilde{U}}$. By using the standard partition of unity construction, we can patch all such

local invariant metrics together into a global metric tensor field of type $(0, 2)$ on the V-manifold M , which we call a Riemannian metric on M .

Let M^m be a compact V-manifold of dimension m with $\mathbb{C}^\infty$ Riemannian metric g , and N^n a $(\mathbb{C}^\infty)$ Riemannian manifold of dimension n . By Satake [12, 13], M admits a finite triangulation $T = \cup s_\alpha$ such that each s_α is contained in the support U_α of a V-chart $\{\tilde{U}_\alpha, G_\alpha, \pi_\alpha\} \in \mathcal{F}$ on M and is the homeomorphic projection of a regular simplex $\tilde{s}_\alpha$ in $\tilde{U}_\alpha$. For a smooth map $f: M \rightarrow N$, the bi-energy functional of f is defined by

$$E_2(f) = \int_M |\tau(f)|^2 * 1 = \sum \int_{s_\alpha} |\tau(f)|^2 dx_\alpha = \sum \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} |\tau(\tilde{f})|^2 d\tilde{x}_\alpha, \quad (2.2)$$

where $d\tilde{x}_\alpha$ denotes the volume form with respect to the G_α -invariant metric g_{ij} in $\tilde{U}_\alpha$, $\tilde{f}_\alpha: \tilde{U}_\alpha \rightarrow N$ is the G_α -invariant lift of f . The Green's divergence theorem on a compact V-manifold proved in [3] plays an important role in the proofs of both Theorems 2.2 and 3.1.

In order to compute the Euler-Lagrange equation, we consider a one-parameter family of maps $\{f_t\} \in \mathbb{C}^\infty(M, N)$, $t \in I_\epsilon = (-\epsilon, \epsilon)$, $\epsilon > 0$ such that in the V-chart $\{\tilde{U}, G, \pi\} \in \mathcal{F}$ over the support U on M , the G -invariant lift $\tilde{f}_t$ is the endpoint of the segment starting at G -invariant lift $\tilde{f}(\tilde{x})$ determined in length and direction by the vector field $\dot{\tilde{f}}$ along $\tilde{f}$, and such that $\partial \tilde{f}_t / \partial t = 0$ and $\tilde{D}_{\tilde{e}_i} \partial \tilde{f}_t / \partial t = 0$ outside a compact subset of the interior of $\tilde{U}$. Choose $\{e_i\}$ being the local frame of a point p in U on M , and $\{\tilde{e}_i\}$ being the local frame of the lifting point $\tilde{p}$ in $\tilde{U}$. Let $D, D', \tilde{D}, \hat{D}$ be the Riemannian connections along $TM, TN, f^{-1}TN, T^*M \otimes f^{-1}TN$, and $\tilde{D}, \hat{\tilde{D}}$ are the Riemannian connections along $T\tilde{U}, T^*\tilde{U} \otimes f^{-1}TN$ in each $\{\tilde{U}, G, \pi\} \in \mathcal{F}$ over the support U on M . Also, let $\Delta = \tilde{D}_{\tilde{e}_k} \tilde{D}_{\tilde{e}_k} - \tilde{D}_{\tilde{D}_{\tilde{e}_k} \tilde{e}_k}$ be the Laplace operator along the cross section of $f^{-1}TN$ in each $\tilde{U}$, and $V = \partial \tilde{f}_t / \partial t$. We can compute (2.2) directly, and obtain the following result.

LEMMA 2.1.

$$\begin{aligned} \frac{d}{dt} E_2(f_t) &= 2\sum \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \left\langle \tilde{D}_{\tilde{e}_i} \tilde{D}_{\tilde{e}_i} d\tilde{f}_t \left(\frac{\partial}{\partial t} \right) - \tilde{D}_{\tilde{D}_{\tilde{e}_i} \tilde{e}_i} d\tilde{f}_t \left(\frac{\partial}{\partial t} \right), (\tilde{D}_{\tilde{e}_j} d\tilde{f}_t)(\tilde{e}_j) \right\rangle d\tilde{x}_\alpha \\ &\quad + 2\sum \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \left\langle R^N \left(d\tilde{f}_t(\tilde{e}_i), d\tilde{f}_t \left(\frac{\partial}{\partial t} \right) \right) d\tilde{f}_t(\tilde{e}_i), (\tilde{D}_{\tilde{e}_j} d\tilde{f}_t)(\tilde{e}_j) \right\rangle d\tilde{x}_\alpha. \end{aligned} \quad (2.3)$$

THEOREM 2.2. *Let $f: (M, \mathcal{F}) \rightarrow N$ be a smooth map from a compact V-manifold $(M, \mathcal{F})$ into a Riemannian manifold N . Set $V = \partial \tilde{f}_t / \partial t$ then*

$$\frac{d}{dt} \Big|_{t=0} E_2(f_t) = 2\sum \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \left\langle V, \Delta \tau(\tilde{f}) + R^N(d\tilde{f}(\tilde{e}_i), \tau(\tilde{f})) d\tilde{f}(\tilde{e}_i) \right\rangle d\tilde{x}_\alpha. \quad (2.4)$$

PROOF. For every $t \in I_\epsilon$, let

$$\tilde{X} = \left\langle \tilde{D}_{\tilde{e}_i} d\tilde{f}_t \left(\frac{\partial}{\partial t} \right), \tilde{D}_{\tilde{e}_j} d\tilde{f}_t(\tilde{e}_j) \right\rangle \tilde{e}_i, \quad \tilde{Y} = \left\langle d\tilde{f}_t \left(\frac{\partial}{\partial t} \right), \tilde{D}_{\tilde{e}_i} (\tilde{D}_{\tilde{e}_j} d\tilde{f}_t)(\tilde{e}_j) \right\rangle (\tilde{e}_i), \quad (2.5)$$

in each $\{\tilde{U}, \pi, G\} \in \mathcal{F}$ over the support U on M . By computing the divergence of $\tilde{X}$ and $\tilde{Y}$ in each $\tilde{U}$, and applying Green's divergence theorem to the vector field $\tilde{X} - \tilde{Y}$

in each $\tilde{\Delta}$ on the compact manifold M in [3], we have

$$\begin{aligned} & \sum \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \left\langle \left(\tilde{D}_{\tilde{e}_i} \tilde{D}_{\tilde{e}_i} d\tilde{f}_t \right) \left(\frac{\partial}{\partial t} \right) - \left(\tilde{D}_{\tilde{D}_{\tilde{e}_i} \tilde{e}_i} d\tilde{f}_t \right) \left(\frac{\partial}{\partial t} \right), \left(\tilde{D}_{\tilde{e}_j} d\tilde{f}_t \right) (\tilde{e}_j) \right\rangle d\tilde{x}_\alpha \\ &= \sum \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \left\langle d\tilde{f}_t \left(\frac{\partial}{\partial t} \right), \tilde{D}_{\tilde{e}_k} \tilde{D}_{\tilde{e}_k} (\tilde{D}_{\tilde{e}_j} d\tilde{f}_t) (\tilde{e}_j) - \tilde{D}_{\tilde{D}_{\tilde{e}_k} \tilde{e}_k} ((\tilde{D}_{\tilde{e}_j} d\tilde{f}_t) (\tilde{e}_j)) \right\rangle d\tilde{x}_\alpha. \end{aligned} \quad (2.6)$$

By the assumption, $\partial \tilde{f}_t / \partial t = 0$ and $\tilde{D}_{\tilde{e}_i} \partial \tilde{f}_t / \partial t = 0$ outside of the compact subset of the interior of each $\tilde{U}$, and substituting (2.6) into (2.3), we get

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} E_2(f_t) &= 2 \sum \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \left\langle d\tilde{f}_t \left(\frac{\partial}{\partial t} \right), \tilde{D}_{\tilde{e}_k} \tilde{D}_{\tilde{e}_k} (\tilde{D}_{\tilde{e}_j} d\tilde{f}_t) (\tilde{e}_j) \right. \\ &\quad \left. - \tilde{D}_{\tilde{D}_{\tilde{e}_k} \tilde{e}_k} ((\tilde{D}_{\tilde{e}_j} d\tilde{f}_t) (\tilde{e}_j)) \right\rangle d\tilde{x}_\alpha \\ &\quad + 2 \sum \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \left\langle R^N \left(d\tilde{f}_t(\tilde{e}_i), d\tilde{f}_t \left(\frac{\partial}{\partial t} \right) \right) d\tilde{f}_t(\tilde{e}_i), (\tilde{D}_{\tilde{e}_j} d\tilde{f}_t) (\tilde{e}_j) \right\rangle d\tilde{x}_\alpha. \end{aligned} \quad (2.7)$$

Let $t = 0$, and by the symmetry of the Riemannian curvature tensor, we derive (2.4). $\square$

DEFINITION 2.3. A smooth map $f : (M, \mathcal{F}) \rightarrow N$ from a compact V-manifold M into a Riemannian manifold N is biharmonic if and only if

$$\tau_2(\tilde{f}) = \Delta \tau(\tilde{f}) + R^N(d\tilde{f}(\tilde{e}_i), \tau(\tilde{f})) d\tilde{f}(\tilde{e}_i) = 0 \quad (2.8)$$

in each $\{\tilde{U}, G, \pi\} \in \mathcal{F}$ over the support U on M .

A harmonic map $f : M \rightarrow N$ on a V-manifold M is obviously a biharmonic map, but a harmonic map is not necessarily a biharmonic map. However, we obtain the following theorem.

THEOREM 2.4. Suppose that M is a compact V-manifold, and N is a Riemannian manifold of nonpositive curvature. If $f : M \rightarrow N$ is a biharmonic map, then f is a harmonic map.

PROOF. In each V-chart $\{\tilde{U}, G, \pi\} \in \mathcal{F}$ over the support U on M it is calculated by

$$\begin{aligned} \Delta e_2(\tilde{f}) &= \frac{1}{2} \Delta \|\tau(\tilde{f})\|^2 = \left\langle \tilde{D}_{\tilde{e}_k} \tau(\tilde{f}), \tilde{D}_{\tilde{e}_k} \tau(\tilde{f}) \right\rangle + \left\langle \tilde{D}^* \tilde{D} \tau(\tilde{f}), \tau(\tilde{f}) \right\rangle \\ &= \left\langle \tilde{D}_{\tilde{e}_k} \tau(\tilde{f}), \tilde{D}_{\tilde{e}_k} \tau(\tilde{f}) \right\rangle - \left\langle R^N(d\tilde{f}(\tilde{e}_i), \tau(\tilde{f})) d\tilde{f}(\tilde{e}_i), \tau(\tilde{f}) \right\rangle \geq 0, \end{aligned} \quad (2.9)$$

because $\tau_2(\tilde{f}) = 0$ in each $\tilde{U}$ and the Riemannian curvature of N is nonpositive. By Bochner's technique and the assumption $\partial \tilde{f}_t / \partial t = 0$ and $\tilde{D}_{\tilde{e}_i} \partial \tilde{f}_t / \partial t = 0$ outside a compact subset of $\text{int}(\tilde{U})$, we know $\|\tau(\tilde{f})\|^2 = \text{const}$, and then substituting into (2.9) we have $\tilde{D}_{\tilde{e}_k}(\tau \tilde{f}) = 0$, for all $k = 1, 2, \dots, m$ by [7] which implies $\tau(\tilde{f}) = 0$ in each $\tilde{U}$, that is, f is harmonic on M . $\square$

Since harmonic maps are automatically biharmonic maps when the Riemannian curvature of N is nonpositive, we will find a non-trivial biharmonic map into a sphere. By the concepts of V-manifolds and the similar techniques as [11], we have the following theorem.

THEOREM 2.5. *Let $f : (M, \mathcal{F}) \rightarrow S^{m+1}$ be nonzero parallel mean curvature isometric embedding, then f is biharmonic if and only if the second fundamental form $B(\tilde{f})$ of $\tilde{f}$ with $\|B(\tilde{f})\|^2 = m = \dim(\tilde{U})$ in each $\tilde{U}$ over the support U on M .*

EXAMPLE 2.6. In S^{m+1} , the compact hypersurface of its Gauss map being isometric embedding is the Clifford surface (see [15]):

$$M_k^m(1) = S^k\left(\sqrt{\frac{1}{2}}\right) \times S^{m-k}\left(\sqrt{\frac{1}{2}}\right), \quad 0 \leq k \leq m. \quad (2.10)$$

Let $f : M_k^m(1) \rightarrow S^{m+1}$ be the standard embedding. Set

$$M_k^m(1)' = \frac{S^k(\sqrt{1/2})}{Z_p} \times \frac{S^{m-k}(\sqrt{1/2})}{Z_{p'}}, \quad (2.11)$$

where p, p' are prime numbers (p and p' could be the same or different). Since both the first and the second terms are compact V-manifolds, the product is also a compact V-manifold. Let $f' : M_k^m(1)' \rightarrow S^{m+1}$ be a map such that $k \neq m/2$, pick $\tilde{U} = \{(x^0, x^1, \dots, x^k) \in S^k\sqrt{1/2} : x^i > 0, i \text{ is any of } 0, 1, \dots, k\} \times \{(x^{k+1}, \dots, x^{m+1}) \in S^{m-k}\sqrt{1/2} : x^j > 0, j \text{ is any of } k+1, \dots, m+1\}$ (if x^i and x^j vary, $\tilde{U}$ is different), and let $\tilde{f}' : \tilde{U} \rightarrow S^{m+1}$ (as part of the standard map $f : S^k\sqrt{1/2} \times S^{m-k}\sqrt{1/2} \rightarrow S^{m+1}$) in each $\{\tilde{U}, G, \pi\} \in \mathcal{F}$. So $\tilde{f}'$ has parallel second fundamental form, and has parallel mean curvature and $B(\tilde{f}') = k + m - k = m, \|\tau(\tilde{f}')\| = |k - (m - k)| = 2k - m \neq 0$. That is, $\tilde{f}'$ is biharmonic in $\tilde{U}$ for each $\{\tilde{U}, G, \pi\} \in \mathcal{F}$. Then by [Theorem 2.5](#) f is a nontrivial biharmonic map on $(M, \mathcal{F})$.

3. The stability and composition of biharmonic maps on V-manifolds. Let M be a compact V-manifold, and N a Riemannian manifold. We continue to use the notations as in the previous sections. By applying the Green's divergence theorem on the compact V-manifold M [3], the concepts of V-manifolds, and the similar techniques in [11], we can have the second variations of biharmonic maps as follows.

THEOREM 3.1. *If $f : (M, \mathcal{F}) \rightarrow N$ is a biharmonic map, then*

$$\begin{aligned} & \frac{1}{2} \frac{d^2}{dt^2} E_2(f_t) \Big|_{t=0} \\ &= \sum \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \|\Delta V + R^N(d\tilde{f}(\tilde{e}_i), V)d\tilde{f}(\tilde{e}_i)\|^2 d\tilde{x}_\alpha \\ &+ \sum \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \langle V, (D'_{d\tilde{f}(\tilde{e}_k)} R^N)(d\tilde{f}(\tilde{e}_k), \tau(\tilde{f}))V \\ &\quad + (D'_{\tau(\tilde{f})} R^N)(d\tilde{f}(\tilde{e}_i), V)d\tilde{f}(\tilde{e}_i) + R^N(\tau(\tilde{f}), V)\tau(\tilde{f}) \\ &\quad + 2R^N(d\tilde{f}(\tilde{e}_k), V)\tilde{D}_{\tilde{e}_k}\tau(\tilde{f}) + 2R^N(d\tilde{f}(\tilde{e}_i), \tau(\tilde{f}))\tilde{D}_{\tilde{e}_i}V \rangle d\tilde{x}_\alpha. \end{aligned} \quad (3.1)$$

DEFINITION 3.2. Let $f : (M, \mathcal{F}) \rightarrow N$ be a biharmonic map from a compact V-manifold M into a Riemannian manifold N . If $d^2/dt^2 E_2(f_t)|_{t=0} \geq 0$, then f is a *stable* biharmonic map.

If we look at a harmonic map as a biharmonic map, then it must be stable by the definition of bi-energy since

$$\frac{1}{2} \frac{d^2}{dt^2} E_2(f_t) \Big|_{t=0} = \sum \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \|\Delta V + R^N(d\tilde{f})((\tilde{e}_i), V) d\tilde{f}(\tilde{e}_i)\|^2 d\tilde{x}_\alpha \geq 0. \quad (3.2)$$

THEOREM 3.3. *Let $f : (M, \mathcal{F}) \rightarrow N$ be a stable biharmonic map from a compact V-manifold M into a Riemannian manifold N of constant sectional curvature $K > 0$ and f satisfies the conservation law, then f must be a harmonic map.*

PROOF. Because N has the constant sectional curvature, the term of $D'R^N$ of the second variation formula disappears and

$$\begin{aligned} \frac{1}{2} \frac{d^2}{dt^2} E_2(f_t) \Big|_{t=0} &= \sum \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \|\Delta V + R^N(df(e_i), V) df(e_i)\|^2 d\tilde{x}_\alpha \\ &\quad + \sum \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \langle V, R^N(\tau(\tilde{f}), V) \tau(\tilde{f}) + 2R^N(d\tilde{f}(\tilde{e}_k), V) \bar{D}_{\tilde{e}_k} \tau(\tilde{f}) \\ &\quad \quad \quad + 2R^N(d\tilde{f}(\tilde{e}_i), \tau(\tilde{f})) \bar{D}_{\tilde{e}_i} V \rangle d\tilde{x}_\alpha. \end{aligned} \quad (3.3)$$

Take $V = \tau(\tilde{f})$, and notice that f is biharmonic and N has the constant sectional curvature, then by (3.3) we have

$$\begin{aligned} \frac{1}{2} \frac{d^2}{dt^2} E_2(f_t) \Big|_{t=0} &= \sum \frac{4}{|G_\alpha|} \int_{\tilde{s}_\alpha} \langle R^N(d\tilde{f}(\tilde{e}_i), \tau(\tilde{f})) \bar{D}_{\tilde{e}_k} \tau(\tilde{f}), \tau(\tilde{f}) \rangle d\tilde{x}_\alpha \\ &= 4K \sum \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \left[\langle d\tilde{f}(\tilde{e}_k), \bar{D}_{\tilde{e}_k} \tau(\tilde{f}) \rangle \|\tau(\tilde{f})\|^2 \right. \\ &\quad \quad \quad \left. - \langle d\tilde{f}(\tilde{e}_k), \tau(\tilde{f}) \rangle \langle \tau(\tilde{f}), \bar{D}_{\tilde{e}_k} \tau(\tilde{f}) \rangle \right] d\tilde{x}_\alpha. \end{aligned} \quad (3.4)$$

In each $\tilde{U}_\alpha$, $\tilde{f}$ satisfies the conservation law [2], so

$$\begin{aligned} \langle d\tilde{f}(\tilde{e}_k), \tau(\tilde{f}) \rangle &= 0, \\ \langle d\tilde{f}(\tilde{e}_k), \bar{D}_{\tilde{e}_k} \tau(\tilde{f}) \rangle &= -\langle \bar{D}_{\tilde{e}_k} d\tilde{f}(\tilde{e}_k), \tau(\tilde{f}) \rangle = -\|\tau(\tilde{f})\|^2 \end{aligned} \quad (3.5)$$

in each $\tilde{U}$. Substitute (3.5) into (3.4), and f is stable, we have

$$\frac{1}{2} \frac{d^2}{dt^2} E_2(f_t) \Big|_{t=0} = -4K \sum \frac{1}{|G_\alpha|} \int_{\tilde{s}_\alpha} \|\tau(\tilde{f})\|^4 d\tilde{x}_\alpha \geq 0. \quad (3.6)$$

Therefore, $\tau(\tilde{f}) = 0$ in each $\tilde{s}_\alpha$ of $\tilde{U}_\alpha$, that is, f is harmonic on $(M, \mathcal{F})$.

Let $f : (M, \mathcal{F}) \rightarrow M'$ be a smooth map from a compact V-manifold $(M, \mathcal{F})$ into a Riemannian manifold and M' , and $f_1 : M' \rightarrow M''$ a smooth map from M' into another Riemannian manifold M'' . Then the composition $f_1 \circ f : M \rightarrow M''$ is a smooth map. Let $D, D', \bar{D}, \bar{D}'\bar{D}, \hat{D}', \hat{D}''$ be the Riemannian connections on $TM, TM', f^{-1}TM, f_1^{-1}TM'', (f_1 \circ f)^{-1}TM'', T^*M \otimes f^{-1}TM', T^*M' \otimes f_1^{-1}TM'', T^*M \otimes (f_1 \circ f)^{-1}TM''$, respectively, and let $R^{M'}(\cdot, \cdot), R^{f_1^{-1}TM''}$ be the Riemannian curvatures on $TM'', f^{-1}TM''$, respectively. For all $X, Y \in \Gamma(TM)$, we have

$$\bar{D}_X'' d(f_1 \circ f)Y = \hat{D}'_{df(X)} df_1(Y) + df_1 \circ \bar{D}_X df(Y). \quad (3.7)$$

□

THEOREM 3.4. *Let $(M, \mathcal{F})$ be a compact V-manifold, and M', M'' Riemannian manifolds. If $f : M \rightarrow M'$ is a biharmonic map and $f_1 : M' \rightarrow M''$ is totally geodesic, then the composition $f_1 \circ f : M \rightarrow M''$ is a biharmonic map.*

PROOF. Since f_1 is totally geodesic, that is, $\hat{D}'df_1 = 0$, so in each $\tilde{U}$ we have $\tau(f_1 \circ \tilde{f}) = df_1 \circ \tau(\tilde{f})$ and

$$\begin{aligned} \tilde{D}''^* \tilde{D}\tau(f_1 \circ \tilde{f}) &= \tilde{D}''^* \tilde{D}''(df_1 \circ \tau(\tilde{f})) \\ &= \tilde{D}_{\tilde{e}_k}'' \tilde{D}_{\tilde{e}_k}''(df_1 \circ \tau(\tilde{f})) - \tilde{D}_{D_{\tilde{e}_k} \tilde{e}_k}''(df_1 \circ \tau(\tilde{f})). \end{aligned} \quad (3.8)$$

By (3.7) and notice that f_1 is totally geodesic, then

$$\begin{aligned} \tilde{D}_{\tilde{e}_k}''(df_1 \circ \tau(\tilde{f})) &= \tilde{D}_{\tilde{e}_k}''(df_1 \circ \hat{D}_{\tilde{e}_j} d\tilde{f}(\tilde{e}_j)) \\ &= (\hat{D}'_{\tilde{D}_{\tilde{e}_j} d\tilde{f}(\tilde{e}_k)} df_1)(\hat{D}_{\tilde{e}_j} d\tilde{f}(\tilde{e}_j)) + df_1 \circ \tilde{D}_{\tilde{e}_k}(\hat{D}_{\tilde{e}_j} d\tilde{f}(\tilde{e}_j)) \\ &= df_1 \circ \tilde{D}_{\tilde{e}_k} \tau(\tilde{f}). \end{aligned} \quad (3.9)$$

So

$$\begin{aligned} \tilde{D}_{\tilde{e}_k}'' \tilde{D}_{\tilde{e}_k}''(df_1 \circ \tau(\tilde{f})) &= \tilde{D}_{\tilde{e}_k}''(df_1 \circ \tilde{D}_{\tilde{e}_k} \tau(\tilde{f})) = df_1 \circ \tilde{D}_{\tilde{e}_k} \tilde{D}_{\tilde{e}_k} \tau(\tilde{f}), \\ \tilde{D}_{D_{\tilde{e}_k} \tilde{e}_k}''(df_1 \circ \tau(\tilde{f})) &= df_1 \circ \tilde{D}_{D_{\tilde{e}_k} \tilde{e}_k} \tau(\tilde{f}). \end{aligned} \quad (3.10)$$

Substituting (3.10) into (3.8), we get

$$\tilde{D}''^* \tau(f_1 \circ \tilde{f}) = df_1 \circ \tilde{D}^* \tilde{D}\tau(\tilde{f}). \quad (3.11)$$

On the other hand,

$$\begin{aligned} R^{M''}(d(f_1 \circ \tilde{f})(\tilde{e}_i), \tau(f_1 \circ \tilde{f})) d(f_1 \circ f)(\tilde{e}_i) \\ = R^{f_1^{-1}TM''}(d\tilde{f}(\tilde{e}_i), \tau(\tilde{f})) df_1(d\tilde{f}(\tilde{e}_i)) \\ = df_1 \circ R^{M'}(d\tilde{f}(\tilde{e}_i), \tau(\tilde{f})) d\tilde{f}(\tilde{e}_i). \end{aligned} \quad (3.12)$$

By (3.11) and (3.12), we have

$$\begin{aligned} \tilde{D}^* \tilde{D}''(f_1 \circ \tilde{f}) + R^{M''}(d(f_1 \circ \tilde{f})(\tilde{e}_i), \tau(f_1 \circ \tilde{f})) d(f_1 \circ \tilde{f})(\tilde{e}_i) \\ = df_1 \circ [\tilde{D}^* \tilde{D}\tau(\tilde{f}) + R^{M'}(d\tilde{f}(\tilde{e}_i), \tau(\tilde{f})) d\tilde{f}(\tilde{e}_i)] \end{aligned} \quad (3.13)$$

in each $\tilde{U}$. Hence, if f is biharmonic, then $f_1 \circ f$ is also biharmonic. $\square$

REMARK 3.5. Theorem 3.4 generalizes the main theorem in [14] into V-manifolds. The condition of f_1 being totally geodesic cannot be weakened into harmonic or biharmonic.

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