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PUTNAM-FUGLEDE THEOREM AND THE RANGE-KERNEL ORTHOGONALITY OF DERIVATIONS

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ABSTRACT. Let $\mathcal{B}(H)$ denote the algebra of operators on a Hilbert space H into itself. Let $d = \delta$ or $\triangle$, where $\delta_{AB} : \mathcal{B}(H) \to \mathcal{B}(H)$ is the generalized derivation $\delta_{AB}(S) = AS - SB$ and $\triangle_{AB} : \mathcal{B}(H) \to \mathcal{B}(H)$ is the elementary operator $\triangle_{AB}(S) = ASB - S$. Given $A, B, S \in \mathcal{B}(H)$, we say that the pair (A,B) has the property $\mathrm{PF}(d(S))$ if $d_{AB}(S) = 0$ implies $d_{A^*B^*}(S) = 0$. This paper characterizes operators A, B, and S for which the pair (A,B) has property $\mathrm{PF}(d(S))$, and establishes a relationship between the $\mathrm{PF}(d(S))$-property of the pair (A,B) and the range-kernel orthogonality of the operator d_{AB}.

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1. Introduction. Let H be a complex Hilbert space, and let $\mathcal{B}(H)$ denote the algebra of operators (i.e., bounded linear transformations) on H into itself. Given $A, B \in \mathcal{B}(H)$, the (classical) Putnam-Fuglede commutativity theorem says that if A, B are normal operators, and if X is an operator such that $AX = XB$, then $A^*X = XB^*$ [9, page 104]. Various generalizations of the Putnam-Fuglede theorem (henceforth shortened to PF-theorem) have appeared over the past three decades (see [4, 8, 13, 14, 15, 17] and some of the references cited in these papers). A generalization of the PF-theorem is obtained when the normality of A and B is replaced by a weaker requirement, such as A and B^* are subnormal or hyponormal. (Here the hypotheses on A, B are asymmetric: there exist subnormal operators A and B, and operators X, such that $AX = XB$ but $A^*X \neq XB^*$ [9, Problem 199, page 107].) Another such generalization of the PF-theorem, considered recently by Okuyama and Watanabe [14], is where the requirement that A and B be normal is removed by requiring more of the intertwining operator X.

Let $\delta_{AB} : \mathcal{B}(H) \to \mathcal{B}(H)$ $(\delta_{AA} = \delta_A)$ denote the generalized derivation $\delta_{AB}(X) = AX - XB$, and let $\triangle_{AB} : \mathcal{B}(H) \to \mathcal{B}(H)$ $(\triangle_{AA} = \triangle_A)$ denote the elementary operator $\triangle_{AB}(X) = AXB - X$. Let $\ker(Y)$ denote the kernel of the operator Y. The (classical) PF-theorem then says that if A, B are normal, then $\ker(\delta_{AB}) = \ker(\delta_{A^*B^*})$. There is a natural $\triangle_{AB}$ analogue, namely that if A, B are normal, then $\ker(\triangle_{AB}) = \ker(\triangle_{A^*B^*})$. Let d denote either δ or $\triangle$. We say that the pair of operators (A,B) has the property $\mathrm{PF}(d)$ (the property $\mathrm{PF}(d(S))$) if $\ker(d_{AB}) \subseteq \ker(d_{A^*B^*})$ (resp., if, given $S \in \mathcal{B}(H)$, $S \in \ker(d_{AB})$ implies $S \in \ker(d_{A^*B^*})$). It is then known that the pair (A,B) has the $\mathrm{PF}(d)$ property for A, B^* belonging to a number of the commonly considered classes of operators (see [15, Theorem 3] and [8, Theorem 2]).

This paper explores the relationship between the range-kernel orthogonality of the operator d_{AB} and the $\mathrm{PF}(d(S))$ property. Recall here that the element x of a normed linear space $\mathcal{V}$, with norm $\|\cdot\|$, is said to be orthogonal to $y \in \mathcal{V}$ if $\|x - \lambda y\| \geq \|\lambda y\|$

for all complex numbers λ. Let the operator S have the polar decomposition $S = U|S|$; suppose that S belongs to the Schatten p-class $\mathscr{C}_p$ for some $1 < p < \infty$. We prove that: $\min\{\|d_{AB}(X)+S\|_p, \|d_{A^*B^*}(X)+S\|_p\} \ge \|S\|_p$ for all $X \in \mathscr{C}_p$ if and only if $S \in \ker(d_{AB})$ and $(A,B) \in \mathrm{PF}(d(S))$ if and only if $d_{AB}(U) = 0 = d_{A^*B^*}(U)$ and $\min\{\|\delta_A(X)+|S^*|\|_p, \|\delta_B(X)+|S|\|_p\} \ge \|S\|_p$ for all $X \in \mathscr{C}_p$ (cf. [7, Theorem]). An analogue of this result is proved for the case in which S is trace class and either S or S^* is injective. We also prove that if A is an isometry such that $\delta_A(S) = 0$ (A is a contraction such that $\triangle_A(S) = 0$), then $\min\{\|\delta_A(T)+S\|, \|\delta_{A^*}(T)+S\|\} \ge \|S\|$ (resp., $\min\{\|\triangle_A(T)+S\|, \|\triangle_{A^*}(T)+S\|\} \ge \|S\|$) for all $T \in \mathscr{B}(H)$. Furthermore, if $S \in \mathscr{B}(H)$ is a *smooth point*, then there exists a rank one operator X such that $\delta_A(X) = 0 = \delta_{A^*}(X)$ (resp., $\triangle_A(X) = 0 = \triangle_{A^*}(X)$). We start (see Section 2) by proving that the pair (A,B) has the $\mathrm{PF}(d(S))$ property if and only if $|S|$ commutes with B, $|S^*|$ commutes with A and $d_{AB}(U) = 0$, where the partial isometry U is as in the polar decomposition $S = U|S|$. This generalizes the result(s) on pairs (A,B) having the $\mathrm{PF}(d)$ property and the result of Okuyama and Watanabe [14].

2. Characterizing pairs $(A,B) \in \mathrm{PF}(d(S))$. In addition to the notation already introduced, we will use the following further notation. The closure of the range of an operator X will be denoted by $\overline{\mathrm{ran}\,X}$. The restriction of X to an invariant subspace M will be denoted by $X|_M$, and the commutator $AB-BA$ of the operators A, B will be denoted by $[A,B]$. The spectrum, the point spectrum, and the approximate point spectrum of X will be denoted by $\sigma(X)$, $\sigma_0(X)$, and $\sigma_a(X)$, respectively. The *trace functional* will be denoted by tr. Recall that a (completely nonunitary) contraction A is said to be of the class $C_{.0}$ of contractions if $\|A^{*n}x\| \to 0$ as $n \to \infty$ for all $x \in H$. Any other notation will be explained as and when required.

The following theorem characterizes pairs of operators (A,B) with the $\mathrm{PF}(d(S))$ property and is the main result of this section.

Theorem 2.1. *Let $A,B,S \in \mathscr{B}(H)$, where S has the polar decomposition $S = U|S|$. Then the pair $(A,B) \in \mathrm{PF}(d(S))$ if and only if*

(i) $[A,|S^*|] = 0$;

(ii) $[B,|S|] = 0$;

(iii) $d_{AB}(U) = 0$.

Proof. We start by considering the case in which $d = \delta$. If $S \in \ker(\delta_{AB})$ and $(A,B) \in \mathrm{PF}(\delta(S))$, then $\delta_{AB}(S) = 0 = \delta_{A^*B^*}(S)$, and so

$$\begin{aligned} A|S^*|^2 &= (AS)S^* = S(BS^*) = SS^*A = |S^*|^2A; \\ B|S|^2 &= (BS^*)S = S^*(AS) = S^*SB = |S|^2B. \end{aligned} \tag{2.1}$$

This implies (i) and (ii). Also, since $\delta_{AB}(S) = 0$ and $[B,|S|] = 0$, $\delta_{AB}(U)|_{\ker^\perp S} = 0$. Clearly, $B : \ker S (= \ker U) \to \ker S$; hence $\delta_{AB}(U) = 0$. Conversely, (ii) and (iii) together imply that $\delta_{AB}(S) = 0$. Since $\overline{\mathrm{ran}\,S}$ reduces A (by (i)) and $\ker^\perp S$ reduces B (by (ii)), it follows from $\delta_{AB}(S) = 0$ that $\delta_{A_1B_1}(S_1) = 0$, where $A_1 = A|_{\overline{\mathrm{ran}\,S}}$, $B_1 = B|_{\ker^\perp S}$ and the quasi-affinity $S_1 : \ker^\perp S \to \overline{\mathrm{ran}\,S}$ is defined by setting $S_1x = Sx$ for each $x \in \ker^\perp S$. Let S_1 have the polar decomposition $S_1 = U_1|S_1|$; then U_1 is a unitary and $|S_1|$ is a quasi-affinity. Clearly, $[B_1,|S_1|] = 0$; hence $\delta_{A_1B_1}(S_1) = 0$ implies that $\delta_{A_1B_1}(U_1) = 0$, that is,

$B_1 = U_1^* A_1 U_1$. Thus $B_1^*|S_1| = |S_1|B_1^*$ implies $U_1^* A_1^* U_1|S_1| = |S_1|B_1^*$, or, $\delta_{A_1^* B_1^*}(S_1) = 0$. This implies that $\delta_{A^* B^*}(S) = 0$.

Now let $d = \triangle$. If $S \in \ker(\triangle_{AB})$ and $(A,B) \in \mathrm{PF}(\triangle(S))$, then $\triangle_{A^0 B^0}(S^0) = 0 = \triangle_{A^{0*} B^{0*}}(S^0)$, $\overline{\operatorname{ran} S^0}$ reduces A^0 and $\ker^\perp S^0$ reduces B^0. (Here X^0 denotes the Berberian extension of the operator X to a Hilbert space $H^0 \supset H$: recall that given a Hilbert space H and an $X \in \mathcal{B}(H)$, there exists a Hilbert space $H^0 \supset H$ and an isometric $*$-isomorphism $X \to X^0$ preserving order such that $\sigma(X^0) = \sigma(X)$, $\sigma_a(X) = \sigma_a(X^0) = \sigma_0(X^0)$ [18, page 15].) Let $A_1 = A^0|_{\overline{\operatorname{ran} S^0}}$, $B_1 = B^0|_{\ker^\perp S^0}$, and let $S_1 : \ker^\perp S^0 \to \overline{\operatorname{ran} S^0}$ denote the quasi-affinity defined by setting $S_1 y = S^0 y$ for each $y \in \ker^\perp S^0$. Then $\triangle_{A_1 B_1}(S_1) = 0$. As stated above, $\sigma_a(B^0) = \sigma_0(B^0)$; hence, since S_1 is a quasi-affinity, $0 \notin \sigma(B_1)$. We have

$$
\begin{aligned}
\triangle_{A_1 B_1}(S_1) = 0 = \triangle_{A_1^* B_1^*}(S_1) &\Longrightarrow \delta_{A_1 B_1^{-1}}(S_1) = 0 = \delta_{A_1^* B_1^{*-1}}(S_1) \\
&\Longrightarrow [A_1, |S_1^*|] = 0 = [B_1, |S_1|] \\
&\Longrightarrow [A^0, |S^{0*}|] = 0 = [B^0, |S^0|] \\
&\Longrightarrow [A, |S^*|] = 0 = [B, |S|],
\end{aligned}
\tag{2.2}
$$

where the second implication follows from the one before by the $d = \delta$ case. To prove (iii), we note that

$$
ASB = S \Longrightarrow AU|S|B = AUB|S| = U|S| \Longrightarrow \triangle_{AB}(U)|_{\ker^\perp S} = 0. \tag{2.3}
$$

Since $B : \ker S \to \ker S$, we conclude that $\triangle_{AB}(U) = 0$. To prove the sufficiency of the conditions, we note that (ii) and (iii) imply that $S \in \ker(\triangle_{AB})$. As before, let $A_1 = A|_{\overline{\operatorname{ran} S}}$, $B_1 = B|_{\ker^\perp S}$ and let $S_1 : \ker^\perp S \to \overline{\operatorname{ran} S}$ be the quasi-affinity defined by setting $S_1 x = Sx$ for each $x \in \ker^\perp S$. Then (by (i) and (ii)) $\triangle_{A_1 B_1}(S_1) = 0$, and $[A_1, |S_1^*|] = 0 = [B_1, |S_1|]$. Let S_1 have the polar decomposition $S_1 = U_1|S_1|$; U_1 unitary. Then $\triangle_{A_1 B_1}(U_1) = 0$, $A_1 U_1 B_1$ (in particular, B_1) is invertible and $A_1 = U_1 B_1^{-1} U_1^*$. We have

$$
A_1^* |S_1^*| = |S_1^*| A_1^* = |S_1^*| U_1 B^{*-1} U_1^* = U_1 |S_1| B^{*-1} U_1^*. \tag{2.4}
$$

Hence $A_1^* S_1 = A_1^* |S_1^*| U_1 = S_1 B^{*-1}$, or, $\triangle_{A_1^* B_1^*}(S_1) = 0$. This implies that $\triangle_{A^* B^*}(S) = 0$, and the proof is complete. □

Remark 2.2. The hypothesis $(A,B) \in \mathrm{PF}(d(S))$ does not imply that $[A,S] = 0$ (or $[B,S] = 0$, or $[A,|S|] = 0$, or $[B,|S^*|] = 0$) for $S \in \ker(d_{AB})$. Thus let U be the (forward) unilateral shift and let

$$
A = U \oplus 1, \qquad B = 1 \oplus U^*, \qquad S = \begin{bmatrix} 0 & 0 \\ (1 - UU^*) & 0 \end{bmatrix}, \tag{2.5}
$$

on $\hat{H} = H \oplus H$. Then A, B^* are subnormal, $S \in \ker(d_{AB})$ and $d_{AB}(S) = 0 = d_{A^* B^*}(S)$. It is easily verified that (i), (ii), and (iii) of Theorem 2.1 are satisfied, but $d_A(S) = -S = d_B(S)(\neq 0)$ and $d_A(|S|) \neq 0 \neq d_B(|S^*|)$. The hypotheses $(A,A) \in \mathrm{PF}(d)$ and $(B,B) \in \mathrm{PF}(d)$ for a class of operators S in $\ker(d_{AB})$ do not guarantee $(A,B) \in \mathrm{PF}(d(S))$. Thus, let $\mathfrak{D}$ denote the closed unit disc in the complex plane, let A be the operator of multiplication by z on $\mathcal{L}^2(\mathfrak{D})$ into itself and let B be the unilateral shift (on a separable Hilbert space H with an orthonormal basis $\{e_n\}_{n\geq 1}$). Then the only compact operator X such that $\delta_B(X) = 0$ is the zero operator (and, trivially, $(B,B) \in \mathrm{PF}(\delta|_{\mathcal{K}(H)})$, where $\mathcal{K}(H)$ is the ideal of compact operators). Define $S : H \to H$ by $(Se_n)(z) = z^n\chi_{\mathfrak{D}_\alpha}$, where $\mathfrak{D}_\alpha = z : |z| \leq \alpha < 1$ for some fixed α. Then $S \in \mathscr{C}_p$ for all $1 \leq p < \infty$, and so is, in particular, compact. (Notice that $\mathrm{tr}(|S|^{2p}) = \sum_{n=1}^{\infty} \int_{\mathfrak{D}_\alpha} |z|^{2np}\,dz = 2\pi\alpha\sum_{n=1}^{\infty}(\alpha^{2np})/(2\mathrm{np}+1) < \infty$.) Clearly, $\delta_{AB}(S) = 0$, but $\delta_{A^*B^*}(S) \neq 0$: for if it were so, then we would have that B has a nontrivial unitary direct summand. Again, the hypotheses $d_{AB}(S) = 0 = d_{A^*B^*}(S)$ do not imply that $A|_{\overline{\mathrm{ran}\,S}}$ and $B|_{\ker^\perp S}$ are normal operators: some additional hypothesis, for example $\ker(d_A) \subseteq \ker(d_{A^*})$ (or $\ker(d_B) \subseteq \ker(d_{B^*})$, or more generally, $\ker(d_{AB}) \subseteq \ker(d_{A^*B^*})$), is required. We note here that if $\ker(d_A) \subseteq \ker(d_{A^*})$, then $AA|S^*| = A|S^*|A$ implies $A^*A|S^*| = A|S^*|A^* = AA^*|S^*|$ implies $A_1 = A|_{\overline{\mathrm{ran}\,S}}$ is normal. Since B_1 $(= B|_{\ker^\perp S})$ in the case in which $d = \delta$ and B_1^{-1} in the case in which $d = \triangle$ is unitarily equivalent to A_1 (see the proof of Theorem 2.1), B_1 is also normal. We remark here that if $d = \triangle$, and A,B are contractions (or, $d = \delta$, A is a contraction and B is invertible with B^{-1} a contraction), then $(A,B) \in \mathrm{PF}(\triangle|_{\mathcal{K}(H)})$ (resp., $(A,B) \in \mathrm{PF}k(\delta|_{\mathcal{K}(H)})$), this follows from [5, Theorem 8 and Corollary 6.4] or [6, Theorem 2(b)].

A well-known result of Barría [3, Lemma 2] says that if V_1 and V_2 are isometries such that $\delta_{V_1^*}(V_2) = 0$, then $\delta_{V_1}(V_2) = 0$. This (indeed more) follows from our theorem, as the following argument shows. It is clear that hypotheses (ii) and (iii) of the theorem are satisfied (with $A = B = V_1^*$ and $S = V_2$). Since $V_2^*V_1 = V_1V_2^*$, $V_2^*V_1V_2 = V_1$, or, $V_1^*V_2^*V_1V_2 = 1 = V_2^*V_1^*V_2V_1$. Also

$$\begin{aligned} \left\|(V_2V_1V_2^* - V_1V_2V_2^*)x\right\|^2 &= 2\left\|V_2^*x\right\|^2 - 2\Re\left(V_2^*V_1^*V_2V_1V_2^*x, V_2^*x\right) \\ &= 2\left\|V_2^*x\right\|^2 - 2\Re\left(V_2^*x, V_2^*x\right) = 0 \end{aligned} \tag{2.6}$$

for all $x \in H$. Hence, $V_2V_1V_2^* = V_1V_2V_2^*$ and $V_2V_2^*V_1 = V_2V_1V_2^* = V_1V_2V_2^*$, that is, (i) of the theorem is satisfied.

Notice that $V_2^*V_1 = V_1V_2^*$ implies $\triangle_{V_2^*V_2}(V_1) = 0$, and the argument above shows that $\triangle_{V_2V_2^*}(V_1) = 0$ also. Indeed our Theorem 2.1 generalizes a recent extension by Okuyama and Watanabe [13, Theorem] of the result of [3], as the following corollary shows.

Corollary 2.3 (see [13, Theorem]). *Let $A,B \in \mathcal{B}(H)$, and let C be a partial isometry such that* (i) $d_{AB}(C) = 0$; (ii) $\|B\| \geq \|A\|$; (iii) $[B,|C|] = 0$; *and* (iv) $C(\|B\|^2 - BB^*)^{1/2} = 0$. *Then $d_{A^*B^*}(C) = 0$.*

Proof. With the partial isometry C replacing the operator S, it is clear that hypotheses (ii) and (iii) of Theorem 2.1 are satisfied. To complete the proof we have to show that $[A,|C^*|] = 0$.

Dividing suitably (if it needs to), we may assume that $\|B\| = 1$; then A is a contraction. Since $C = C|B^*|^2$ (by hypothesis (iv) above), $\delta_{AA^*}(|C^*|^2) = \delta_{CC^*}(|B^*|^2) = 0$. Now extend the contraction A to a partial isometry $\tilde{A}$, on $\tilde{H} = H \oplus H$ (say), by setting

$$\tilde{A} = \begin{bmatrix} A & (1 - AA^*)^{1/2} \\ 0 & 0 \end{bmatrix} \tag{2.7}$$

(see [9, page 72]), and let $X : \tilde{H} \to \tilde{H}$ be defined by $X = CC^* \oplus 0$. Then $\triangle_{\tilde{A}\tilde{A}^*} = 0$, where

$\widetilde{A}$ being a partial isometry, has $C_{.0}$ completely nonunitary part. Applying [6, Theorem 2(a)], it follows that $\overline{\operatorname{ran} X}$ reduces $\widetilde{A}$ and $\widetilde{A}|_{\overline{\operatorname{ran} X}}$ is unitary. Hence $\delta_{A^*}(X) = 0$. This implies that $[A, |C^*|^2] = 0$. □

Let d^n_{AB}, $n \geq 1$ some integer, denote an n-times application of d_{AB}. Then $\ker d_{AB} \subseteq \ker d^n_{AB}$ for all $n > 1$; the converse is however false in general. Additional hypotheses on A and B, such as A, B^* are normal or subnormal or hyponormal [8, 15], are required for $\ker d^n_{AB} = \ker d_{AB}$ to hold. An example of classes of operators A, B^* for which $\ker d^n_{AB} = \ker d_{AB}$ has been considered in [8, Lemma 4]; the following corollary generalizes [16, Theorem 1] and [8, Lemma 4]. Let $(A,B) \in \mathrm{PF}[d^r(S)]$, where r is some natural number, denote $d_{AB}(d^r(S)) = 0$ implies $d_{A^*B^*}(d^r(S)) = 0$.

Corollary 2.4. *Given $A, B, S \in \mathcal{B}(H)$, suppose that $(A,B) \in \mathrm{PF}[d^r(S)] \bigcap \mathrm{PF}\,[Ad^r(S)]$ for all $r = 1, 2, \ldots, n-1$. Then $d^n_{AB}(S) = 0$ if and only if $d_{AB}(S) = 0$, $\overline{\operatorname{ran} S}$ reduces A, $\ker^{\perp} S$ reduces B, and $A|_{\overline{\operatorname{ran} S}}$ and $B|_{\ker^{\perp} S}$ are normal operators.*

Proof. We consider the case in which $d = \triangle$; the case $d = \delta$ is similarly dealt with. Let $S \in \ker \triangle^n_{AB}$ and let $X = \triangle^{n-1}_{AB}(S)$. The hypothesis $(A,B) \in \mathrm{PF}[d^{n-1}(S)] \bigcap \mathrm{PF}\,[Ad^{n-1}(S)]$ implies that

$$\begin{aligned} AXB - X = 0 = A^*XB^* - X; \\ A(AX)B - (AX) = 0 = A^*(AX)B^* - (AX), \end{aligned} \tag{2.8}$$

and hence that

$$A^*AXB^* - AX = AA^*XB^* - AX \qquad \text{or} \qquad (A^*A - AA^*)XB^* = 0. \tag{2.9}$$

Since $\overline{\operatorname{ran} X}$ reduces A, $\ker^{\perp} X$ reduces B, and $A_1 = A|_{\overline{\operatorname{ran} X}}$ and $B_1^{-1} = (B|_{\ker^{\perp} X})^{-1}$ are unitarily equivalent (see the proof of Theorem 2.1), it follows that A_1 and B_1 are normal operators. Let $S : \ker^{\perp} X \oplus \ker X \to \overline{\operatorname{ran} X} \oplus (\overline{\operatorname{ran} X})^{\perp}$ have the matrix representation $S = [S_{ij}]^2_{i,j=1}$. Letting $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, it then follows that

$$\triangle^n_{A_1B_1}(S_{11}) = 0, \qquad X = \triangle^{n-1}_{AB}(S) = \left[\triangle^{n-1}_{A_iB_j}(S_{ij})\right]^2_{i,j=1} = \triangle^{n-1}_{A_1B_1}(S_{11}) \oplus 0. \tag{2.10}$$

The operators A_1 and B_1 being normal, $\triangle^n_{A_1B_1}(S_{11}) = 0$ if and only if $\triangle_{A_1B_1}(S_{11}) = 0$; hence $X = 0$. Repeating this argument a finite number of times, with $X = \triangle^{n-1}_{AB}(S)$ replaced by $X = \triangle^{n-2}_{AB}(S)$ and so forth, it now follows that $\triangle_{AB}(S) = 0$, where the operators $A|_{\overline{\operatorname{ran} S}}$ and $B|_{\ker^{\perp} S}$ are normal. □

The conclusions of Corollary 2.4 remain valid if the hypothesis that $(A,B) \in \mathrm{PF}[d^r(S)] \cap \mathrm{PF}[Ad^r(S)]$ is replaced by the hypothesis that $(A,B) \in \mathrm{PF}[d^r(S)] \cap \mathrm{PF}[d^r(S)B]$.

Remark 2.5. Let $\pi : \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$ denote the Calkin map. Let $A, B, S \in \mathcal{B}(H)$ be such that $(\pi(A), \pi(B)) \in \mathrm{PF}[\pi(d^r(S))] \cap \mathrm{PF}[\pi(Ad^r(S))]$ for all $r = 1, 2, \ldots, n-1$, and $d^n_{AB}(S)$ is compact for some integer $n > 1$. Then $\pi(d^n_{AB}(S)) = 0$, and it follows from Corollary 2.4 that $\pi(d_{AB}(S)) = 0$, that is, $d_{AB}(S)$ is compact (cf. [16, Theorem 6] and [8, Remark, page 86]).

3. Range-kernel orthogonality and the PF-property. In this section, we explore the relationship between the range kernel orthogonality of d_{AB} and the PF-property $d_{AB}(S) = 0 = d_{A^*B^*}(S)$. Throughout the following, we assume our Hilbert space H to be separable. The operator S will be said to belong to the Schatten p-class $\mathscr{C}_p = \mathscr{C}_p(H)$, $1 \leq p \leq \infty$, if $\|S\|_p = (\operatorname{tr}|S|^p)^{1/p} < \infty$. The range-kernel orthogonality of d_{AB}, with respect to the norms $\|\cdot\|_p$ and $\|\cdot\|$ (= the usual operator norm), has been considered by a number of authors in the recent past (see [7, 10], and some of the references cited there). A definitive result here is the following proposition. Let S have the polar decomposition $S = U|S|$.

Proposition 3.1. *If $A, B \in \mathscr{B}(H)$, and $S \in \mathscr{C}_p$ for some $1 < p < \infty$, then*

$$\left\|d_{AB}(X) + S\right\|_p \geq \|S\|_p \tag{3.1}$$

for all $X \in \mathscr{C}_p$ if and only if $\operatorname{tr}(|S|^{p-1}U^* d_{AB}(X)) = 0$ for all $X \in \mathscr{C}_p$ if and only if $d_{BA}(|S|^{p-1}U^*) = 0$.

Proof. See [10, Theorem 2] and [7, Lemma 2]. □

Proposition 3.1 has $\|\cdot\|$, $\|\cdot\|_1$ and $\|\cdot\|_\infty$ analogues (see [10, Remarks, page 872] for the case $d = \delta$). Recall that the operator S with $\|S\| = 1$ is said to be a smooth point of the unit ball of $\mathscr{B}(H)$ if $\|\cdot\|$ is Gateaux differentiable at S, that is, if the essential norm $\|S\|_e$ of S satisfies $\|S\|_e < \|S\|$, and if S attains its norm at a unique (up to multiplication by a constant of modulus one) unit vector $f \in H$ [11]. (The space $\mathscr{C}_p$, $1 < p < \infty$, being uniformly convex, every $S \in \mathscr{C}_p$ is a smooth point.) The following analogue of Proposition 3.1 will be required in our considerations below.

Lemma 3.2. *Let $S \in \mathscr{B}(H)$ be a smooth point, and let f be the unique unit vector at which S attains its norm. If $A, B \in \mathscr{B}(H)$, then the following statements are equivalent:*

(i) $\|d_{AB}(X) + S\| \geq \|S\|$ *for all $X \in \mathscr{B}(H)$.*

(ii) $\operatorname{tr}((f \otimes Sf) d_{AB}(X)) = 0$ *for all $X \in \mathscr{B}(H)$.*

(iii) $d_{BA}(f \otimes Sf) = 0$.

Proof. The case $d_A = \delta_A$ is dealt with in [10, Remarks (2), page 872]; the proof of the general case follows from a similar argument (see also the proof of [7, Lemma 2]). □

Theorem 3.3. *Let $S \in \mathscr{B}(H)$ be a smooth point.*

(i) *If V is an isometry such that $\delta_V(S) = 0$, then there exists a rank one operator X such that*

$$\delta_V(X) = 0 = \delta_{V^*}(X). \tag{3.2}$$

(ii) *If A is a contraction such that $\triangle_A(S) = 0$, then there exists a rank one operator X such that*

$$\triangle_A(X) = 0 = \triangle_{A^*}(X). \tag{3.3}$$

The proof of the theorem proceeds through a couple of steps, stated below as lemmas. The first of these lemmas states that if A, B are any contractions such that $\triangle_{AB}(T) = 0$ for some $T \in \mathscr{B}(H)$, then the range of $\triangle_{AB}$ is orthogonal to T. This result is then used in the following lemma to prove (and extend) a result of Anderson

[2, Theorem 1] on the range-kernel orthogonality of δ_V for isometries V. The proof of the theorem is then obtained by appealing to Lemma 3.2.

Lemma 3.4. *Let A, B be contractions such that $\triangle_{AB}(S) = 0$ for some $S \in \mathcal{B}(H)$. Then*

$$||\triangle_{AB}(X) + S|| \geq \|S\| \tag{3.4}$$

for all $X \in \mathcal{B}(H)$.

Proof. The inspiration for the following proof comes from the proof of [2, Theorem 1].

Given $X \in \mathcal{B}(H)$, a simple calculation shows that

$$\sum_{i=0}^{n-1} A^{n-i-1} \triangle_{AB}(X) B^{n-i-1} = A^n X B^n - X. \tag{3.5}$$

Thus, if $S \in \ker(\triangle_{AB})$, then

$$S = -\frac{1}{n}\left\{A^n X B^n - X - \sum_{i=0}^{n-1} A^{n-i-1}\left(\triangle_{AB}(X) + S\right) B^{n-i-1}\right\}. \tag{3.6}$$

Hence

$$\begin{aligned}\|S\| &\leq \frac{1}{n}||A^n X B^n - X|| + \frac{1}{n}\left\{\sum_{i=0}^{n-1} \|A\|^{n-i-1} \|B\|^{n-i-1} ||\triangle_{AB}(X) + S||\right\} \\ &\leq \frac{1}{n}||A^n X B^n - X|| + ||\triangle_{AB}(X) + S||.\end{aligned} \tag{3.7}$$

Letting $n \to \infty$, the proof follows. □

Lemma 3.5. *Let V be an isometry such that $\delta_V(T) = 0$ for some $T \in \mathcal{B}(H)$. Then*

$$\min\left\{||\delta_V(X) + T||, ||\delta_{V^*}(X) + T||\right\} \geq \|T\| \tag{3.8}$$

for all $X \in \mathcal{B}(H)$.

Proof. If $\delta_V(T) = 0$, V is an isometry, then

$$||\delta_V(X) + T|| \geq ||V^*(\delta_V(X) + T)|| = ||-\triangle_{V^*V}(X) + V^*T|| = ||\triangle_{V^*V}(X) - V^*T|| \tag{3.9}$$

for all $X \in \mathcal{B}(H)$. Since $\delta_V(T) = 0$ implies $\triangle_{V^*V}(T) = 0$, we have (upon choosing $A = V^*$, $B = V$ and $S = -V^*T$ in Lemma 3.4) that

$$||\delta_V(X) + T|| \geq ||\triangle_{V^*V}(X) + (-V^*T)|| \geq ||V^*T|| \tag{3.10}$$

for all $X \in \mathcal{B}(H)$. That $\|\delta_V(X) + T\| \geq \|T\|$ for all $X \in \mathcal{B}(H)$ now follows from the fact that

$$\delta_V(T) = 0 \Longrightarrow T = V^*TV \Longrightarrow \|T\| = ||V^*TV|| \leq ||V^*T|| \|V\| = ||V^*T|| \leq \|T\|. \tag{3.11}$$

Again, if $\delta_V(T) = 0$ with V an isometry, then

$$||\delta_{V^*}(X) + T|| \geq ||(\delta_{V^*}(X) + T)V|| = ||\triangle_{V^*V}(X) + TV||, \tag{3.12}$$

and hence, since $V^*TV-T=0$ implies $\triangle_{V^*V}(TV)=0$,

$$||\delta_{V^*}(X)+T|| \geq \|TV\| = \|T\| \tag{3.13}$$

for all $X \in \mathcal{B}(H)$. This completes the proof. □

Results of the type of Lemma 3.4 have been proved earlier, but under the stronger hypothesis that the intertwining operator S is compact (cf. [12]). The argument of the proof of Lemma 3.5 in fact leads to a stronger result, namely that: if A is left invertible by a contraction, the operator B is a contraction, and if $T \in \ker(\delta_{AB})$, then $\|\delta_{AB}(X)+T\| \geq \|T\|$ for all $X \in \mathcal{B}(H)$.

Proof of Theorem 3.3. If V is an isometry such that $\delta_V(S)=0$, then Lemma 3.5 implies that

$$\min\{||\delta_V(X)+S||, ||\delta_{V^*}(X)+S||\} \geq \|S\| \tag{3.14}$$

for all $X \in \mathcal{B}(H)$. Assuming now that S is a smooth point, it follows from Lemma 3.2 that there exists a unique (up to multiplication by a constant of modulus one) unit vector $f \in H$ such that

$$\delta_V(f\otimes Sf)=0=\delta_{V^*}(f\otimes Sf). \tag{3.15}$$

The operator $X = f\otimes Sf$ is then the required rank one operator. Since a similar argument, using this time Lemmas 3.4 and 3.2, implies the existence of a rank one operator X such that $\triangle_A(X)=0$, and since this (in view of the compactness of X) implies by [5, Theorem 8] that $\triangle_{A^*}(X)=0$, the proof is complete. □

The rank one operator X in Theorem 3.3 satisfies $\|X\|_1 = \|S\|$ and $\operatorname{tr}(SX)=\|S\|^2$ (see [10, Lemma 1]). Also, in view of Lemma 3.2, $\delta_V(X)=0=\delta_{V^*}(X)$ if and only if $\min\{\|\delta_V(Y)+S\|, \|\delta_{V^*}(Y)+S\|\} \geq \|S\|$, and $\triangle_A(X)=0=\triangle_{A^*}(X)$ if and only if $\min\{\|\triangle_A(Y)+S\|, \|\triangle_{A^*}(Y)+S\|\} \geq \|S\|$, for all $Y \in \mathcal{B}(H)$.

We consider now the case $d_{AB}|_{\mathcal{C}_p}$, where $A,B \in \mathcal{B}(H)$ and $1<p<\infty$. Recall from Proposition 3.1 that, given $S \in \mathcal{C}_p$,

$$\min\{||d_{AB}(X)+S||_p, ||d_{A^*B^*}(X)+S||\} \geq \|S\|_p \tag{3.16}$$

if and only if

$$d_{BA}(|S|^{p-1}U^*)=0=d_{B^*A^*}(|S|^{p-1}U^*). \tag{3.17}$$

As seen in the proof of Theorem 2.1, (3.17) implies that $|S|^{2(p-1)}$ and so also $|S|$ commutes with B, $|S^*|=U|S|U^*$ commutes with A, and $d_{BA}(U^*)=0=d_{B^*A^*}(U^*)$. Hence $d_{BA}(S^*)=0=d_{B^*A^*}(S^*)$, that is,

$$d_{AB}(S)=0=d_{A^*B^*}(S). \tag{3.18}$$

Thus, given an $S \in \mathcal{C}_p$ $(1<p<\infty)$, (3.16) holds for all $X \in \mathcal{C}_p$ if and only if $S \in \ker(d_{AB})$ and $(A,B) \in \operatorname{PF}(d(S))$ (see also [7]).

Theorem 3.6. *Let $A,B \in \mathcal{B}(H)$, and let $S(=U|S|) \in \mathcal{C}_p$ for some $1<p<\infty$. The following statements are equivalent:*

(i) *Inequality (3.16) holds for all $X \in \mathcal{C}_p$.*

(ii) *$S \in \ker(d_{AB})$ and $(A,B) \in \operatorname{PF}(d(S))$.*

(iii) $d_{AB}(U) = 0 = d_{A^*B^*}(U)$, *and*

$$\min\{||\delta_A(X)+|S^*|||_p, ||\delta_{A^*}(X)+|S^*|||_p, ||\delta_B(X)+|S|||_p, ||\delta_{B^*}(X)+|S|||_p\} \geq \|S\|_p \quad (3.19)$$

for all $X \in \mathscr{C}_p$.

Proof. As seen above, (i)⇔(ii). To prove (ii)⇔(iii), we start by noting that if (ii) holds, then, by Theorem 2.1 and its proof, $[A, |S^*|] = 0 = [B, |S|]$ and $d_{AB}(U) = 0 = d_{A^*B^*}(U)$. Hence to prove that (ii)⇔(iii), it will suffice to prove that inequality (3.19) holds if and only if $[A, |S^*|] = 0 = [B, |S|]$. Let T denote either of $|S|$ and $|S^*|$. Then, since $S \in \mathscr{C}_p$, $T \in \mathscr{C}_p$. The map $T \to \|T\|_p^p$ is Frechèt differentiable, with the Frechèt derivative D_T given by $D_T(Y) = p\Re\operatorname{tr}(T^{p-1}Y)$ (see [1, Theorem 2.1]). Let $Z = A$ or A^* in the case in which $T = |S^*|$ in $\delta_Z(T)$, and let $Z = B$ or B^* in the case in which $T = |S|$ in $\delta_Z(T)$. Then Proposition 3.1 translates to $\|\delta_Z(X) + T\|_p \geq \|T\|_p$ for all $X \in \mathscr{C}_p$ if and only if $(\operatorname{tr}(T^{p-1}\delta_Z(X)) = 0$ for all $X \in \mathscr{C}_p$ if and only if $\delta_Z(T^{p-1}) = 0$ (see [10, Theorem 2] and [7, Lemma 2]). Hence inequality (3.19) holds if and only if $\delta_Z(T) = 0$.

We close this paper by considering the case $d_{AB}|_{\mathscr{C}_1}$. Let $S = U|S| \in \mathscr{C}_1$ be such that either S or S^* is injective. Then S is a smooth point (of $\operatorname{Ball}(\mathscr{C}_1)$) and the map $S \to \|S\|_1$ is Frechèt differentiable. Let $V = U^*$ if S is injective and $V = U$ if S^* is injective. Then

$$\min\{||d_{AB}(X) + S||_1, ||d_{A^*B^*}(X) + S||_1\} \geq \|S\|_1 \quad (3.20)$$

for all $X \in \mathscr{C}_1$ if and only if

$$\operatorname{tr}(Vd_{AB}(X)) = 0 = \operatorname{tr}(Vd_{A^*B^*}(X)) \quad (3.21)$$

for all $X \in \mathscr{C}_1$. (This is proved for the case in which $d = \delta$ and $A = B$ in [10]; the general case follows from a similar argument.) Choose X to be the rank one operator $(x \otimes y)$; $x, y \in H$.Then,since VAX and VX are in $\mathscr{C}_1$ for all $X \in \mathscr{C}_1$,

$$\begin{aligned}
\operatorname{tr}(V\triangle_{AB}(X)) = 0 &\Longleftrightarrow \operatorname{tr}(\triangle_{BA}(V)X) = 0 \Longleftrightarrow (\triangle_{BA}(V)x, y) = 0;\\
\operatorname{tr}(V\triangle_{A^*B^*}(X)) = 0 &\Longleftrightarrow \operatorname{tr}(\triangle_{B^*A^*}(V)X) = 0 \Longleftrightarrow (\triangle_{B^*A^*}(V)x, y) = 0,\\
\operatorname{tr}(V\delta_{AB}(X)) = 0 &\Longleftrightarrow \operatorname{tr}(-\delta_{BA}(V)X) = 0 \Longleftrightarrow (\delta_{BA}(V)x, y) = 0;\\
\operatorname{tr}(V\delta_{A^*B^*}(X)) = 0 &\Longleftrightarrow \operatorname{tr}(-\delta_{B^*A^*}(V)X) = 0 \Longleftrightarrow (\delta_{B^*A^*}(V)x, y) = 0
\end{aligned} \quad (3.22)$$

for all $x, y \in H$. Hence, if (3.20) holds, then

$$d_{AB}(V^*) = 0 = d_{A^*B^*}(V^*). \quad (3.23)$$

□

Theorem 3.7. *Let $S \in \mathscr{C}_1$ be such that either S or S^* is injective. If $A, B \in \mathscr{B}(H)$ and the operator V is as above, then the following statements are equivalent:*

(i) *Inequality (3.20) holds for all $X \in \mathscr{C}_1$.*

(ii) $V^* \in \ker(d_{AB})$ *and* $(A, B) \in \operatorname{PF}(d(V^*))$.

Proof. We have already seen that (i)⇒(ii). To prove that (ii)⇒(i), let $X \in \mathscr{C}_1$. Then both VAX and VX are in $\mathscr{C}_1$. By hypothesis $d_{AB}(V^*) = 0 = d_{BA}(V)$. Hence

$$\begin{aligned}
\operatorname{tr}(V\triangle_{AB}(X)) &= \operatorname{tr}(VAXB) - \operatorname{tr}(VX) = \operatorname{tr}(BVAX) - \operatorname{tr}(VX) = \operatorname{tr}(\triangle_{BA}(V)X) = 0;\\
\operatorname{tr}(V\delta_{AB}(X)) &= \operatorname{tr}(VAX) - \operatorname{tr}(VXB) = \operatorname{tr}(VAX) - \operatorname{tr}(BVX) = \operatorname{tr}(-\delta_{BA}(V)X) = 0.
\end{aligned} \quad (3.24)$$

Since these equalities remain true when A and B are replaced by A^* and B^*, respectively, it follows (from above) that (ii)⇒(i). □

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