ON MATRIX TRANSFORMATIONS CONCERNING
THE NAKANO VECTOR-VALUED
SEQUENCE SPACE

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ABSTRACT. We give the matrix characterizations from Nakano vector-valued sequence space \( \ell(X,p) \) and \( F_r(X,p) \) into the sequence spaces \( E_r, \ell_\infty, \ell_\infty(q), bs, \) and \( cs \), where \( p = (p_k) \) and \( q = (q_k) \) are bounded sequences of positive real numbers such that \( p_k > 1 \) for all \( k \in \mathbb{N} \) and \( r \geq 0 \).

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1. Introduction. Let \( (X, \| \cdot \|) \) be a Banach space, \( r \geq 0 \) and \( p = (p_k) \) a bounded sequence of positive real numbers. We write \( x = (x_k) \) with \( x_k \in X \) for all \( k \in \mathbb{N} \). The \( X \)-valued sequence spaces \( c_0(X,p), c(X,p), \ell_\infty(X,p), \ell(X,p), E_r(X,p), F_r(X,p), \) and \( \ell_\infty(X,p) \) are defined as

\[

c_0(X,p) = \{ x = (x_k) : \lim_{k \to \infty} \|x_k\|^{p_k} = 0 \}, \\
c(X,p) = \{ x = (x_k) : \lim_{k \to \infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X \}, \\
\ell_\infty(X,p) = \{ x = (x_k) : \sup_{k} \|x_k\|^{p_k} < \infty \}, \\
\ell(X,p) = \{ x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty \}, \quad \tag{1.1}
\]

\[
E_r(X,p) = \{ x = (x_k) : \sup_{k} \frac{\|x_k\|^{p_k}}{k^r} < \infty \}, \\
F_r(X,p) = \{ x = (x_k) : \sum_{k=1}^{\infty} k^r \|x_k\|^{p_k} < \infty \},
\]

\[
\ell_\infty(X,p) = \bigcap_{n=1}^{\infty} \{ x = (x_k) : \sup_{k} \|x_k\|^{n^{1/p_k}} \}.
\]

When \( X = K \), the scalar field of \( X \), the corresponding spaces are written as \( c_0(p), c(p), \ell_\infty(p), \ell(p), E_r(p), F_r(p), \) and \( \ell_\infty(p) \), respectively. The spaces \( c_0(p), c(p), \) and \( \ell_\infty(p) \) are known as the sequence spaces of Maddox. These spaces were first introduced and studied by Simons [7] and Maddox [4, 5]. The space \( \ell(p) \) was first defined by Nakano [6] and it is known as the Nakano sequence space and the space \( \ell(X,p) \) is known as the Nakano vector-valued sequence space. When \( p_k = 1 \) for all \( k \in \mathbb{N} \), the spaces \( E_r(p) \) and \( F_r(p) \) are written as \( E_r \) and \( F_r \), respectively. These two
sequence spaces were first introduced by Cooke [1]. The space $\ell_\infty(p)$ was first defined by Grosse-Erdmann [2] and he has given in [3] characterizations of infinite matrices mapping between scalar-valued sequence spaces of Maddox. Wu and Liu [10] gave necessary and sufficient conditions for infinite matrices mapping from $c_0(X,p)$ and $\ell_\infty(X,p)$ into $c_0(q)$ and $\ell_\infty(q)$. Suantai [8] has given characterizations of infinite matrices mapping $\ell(X,p)$ into $\ell_\infty$ and $\ell_\infty(q)$ when $p_k \leq 1$ for all $k \in \mathbb{N}$ and he has also given in [9] characterizations of those infinite matrices mapping from $\ell(X,p)$ into the sequence space $E_r$ when $p_k \leq 1$ for all $k \in \mathbb{N}$.

In this paper, we extend the results of [8, 9] in case $p_k > 1$ for all $k \in \mathbb{N}$. Moreover, we also give the matrix characterizations from $\ell(X,p)$ and $F_r(X,p)$ into the sequence spaces $bs$ and $cs$.

2. Notations and definitions. Let $(X,\| \cdot \|)$ be a Banach space, the space of all sequences in $X$ is denoted by $W(X)$, and $\Phi(X)$ denotes the space of all finite sequences in $X$. When $X = K$, the scalar field of $X$, the corresponding spaces are written as $w$ and $\Phi$.

A sequence space in $X$ is a linear subspace of $W(X)$. Let $E$ be an $X$-valued sequence space. For $x \in E$ and $k \in \mathbb{N}$, $x_k$ stands for the $k$th term of $x$. For $k \in \mathbb{N}$, we denote by $e_k$ the sequence $(0,0,\ldots,0,1,0,\ldots)$ with 1 in the $k$th position and by $e$ the sequence $(1,1,1,\ldots)$. For $x \in X$ and $k \in \mathbb{N}$, let $e_k(x)$ be the sequence $(0,0,\ldots,0,x,0,\ldots)$ with $x$ in the $k$th position and let $e(x)$ be the sequence $(x,x,x,\ldots)$. We call a sequence space $E$ normal if $(t_kx_k) \in E$ for all $x = (x_k) \in E$ and $t_k \in K$ with $|t_k| = 1$ for all $t_k \in \mathbb{N}$. A normed sequence space $(E,\| \cdot \|)$ is said to be norm monotone if $x = (x_k)$, $y = (y_k) \in E$ with $\|x_k\| \leq \|y_k\|$ for all $k \in \mathbb{N}$ we have $\|x\| \leq \|y\|$. For a fixed scalar sequence $\mu = (\mu_k)$, the sequence space $E_\mu$ is defined as

$$E_\mu = \{x \in W(X) : (\mu_kx_k) \in E\}. \quad (2.1)$$

Let $A = (f^n_k)$ with $f^n_k$ in $X'$, the topological dual of $X$. Suppose that $E$ is a space of $X$-valued sequences and $F$ a space of scalar-valued sequences. Then $A$ is said to map $E$ into $F$, written by $A : E \rightarrow F$, if for each $x = (x_k) \in E$, $A_n(x) = \sum_{k=1}^{\infty} f^n_k(x_k)$ converges for each $n \in \mathbb{N}$, and the sequence $Ax = (A_n(x)) \in F$. Let $(E,F)$ denote the set of all infinite matrices mapping from $E$ into $F$.

Suppose that the $X$-valued sequence space $E$ is endowed with some linear topology $\tau$. Then $E$ is called a $K$-space if for each $k \in \mathbb{N}$, the $k$th coordinate mapping $p_k : E \rightarrow X$, defined by $p_k(x) = x_k$, is continuous on $E$. If, in addition, $(E,\tau)$ is a Fréchet (Banach) space, then $E$ is called an FK- (BK-) space. Now, suppose that $E$ contains $\Phi(X)$. Then $E$ is said to have property AB if the set $\{\sum_{k=1}^{\infty} e^k(x_k) : n \in \mathbb{N}\}$ is bounded in $E$ for every $x = (x_k) \in E$. It is said to have property AK if $\sum_{k=1}^{\infty} e^k(x_k) \rightarrow x$ in $E$ as $n \rightarrow \infty$ for every $x = (x_k) \in E$. It has property AD if $\Phi(X)$ is dense in $E$.

It is known that the Nakano sequence space $\ell(X,p)$ is an FK-space with property AK under the paranorm $g(x) = (\sum_{k=1}^{\infty} \|x_k\|^p_k)^{1/M}$, where $M = \max\{1,\sup_k p_k\}$. If $p_k > 1$ for all $k \in \mathbb{N}$, then $\ell(X,p)$ is a BK-space with the Luxemburg norm defined by

$$\|(x_k)\| = \inf \left\{ \varepsilon > 0 : \sum_{k=1}^{\infty} \left\| \frac{x_k}{\varepsilon} \right\|^{p_k} \leq 1 \right\}. \quad (2.2)$$
3. Main results. We first give a characterization of an infinite matrix mapping from \( \ell(X,p) \) into \( E_r \) when \( p_k > 1 \) for all \( k \in \mathbb{N} \). To do this, we need the following lemma.

**Lemma 3.1.** Let \( E \) be an \( X \)-valued BK-space which is normal and norm monotone and let \( A = (f_k^n) \) be an infinite matrix. Then \( A : E \rightarrow E_r \) if and only if \( \sup_n \sum_{k=1}^{\infty} |f_k^n(x_k)|/n^r < \infty \) for every \( x = (x_k) \in E \).

**Proof.** If the condition holds true, it follows that

\[
\sup_n \frac{\sum_{k=1}^{\infty} f_k^n(x_k)}{n^r} \leq \sup_n \sum_{k=1}^{\infty} \frac{|f_k^n(x_k)|}{n^r} < \infty
\]

for every \( x = (x_k) \in E \), hence \( A : E \rightarrow E_r \).

Conversely, assume that \( A : E \rightarrow E_r \). Since \( E \) and \( E_r \) are BK-spaces, by Zeller’s theorem, \( A : E \rightarrow E_r \) is bounded, so there exists \( M > 0 \) such that

\[
\sup_{n \in \mathbb{N}} \frac{\sum_{k=1}^{\infty} f_k^n(x_k)}{n^r} \leq M.
\]

Let \( x = (x_k) \in E \) be such that \( \|x\| = 1 \). For each \( n \in \mathbb{N} \), we can choose a scalar sequence \( (t_k) \) with \( |t_k| = 1 \) and \( f_k^n(t_kx_k) = |f_k^n(x_k)| \) for all \( k \in \mathbb{N} \). Since \( E \) is normal and norm monotone, we have \( (t_kx_k) \in E \) and \( \|(t_kx_k)\| \leq 1 \). It follows from (3.2) that

\[
\sum_{k=1}^{\infty} \frac{|f_k^n(x_k)|}{n^r} = \frac{\sum_{k=1}^{\infty} f_k^n(t_kx_k)}{n^r} \leq M,
\]

which implies

\[
\sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} \frac{|f_k^n(x_k)|}{n^r} \leq M.
\]

It follows from (3.4) that for every \( x = (x_k) \in E \),

\[
\sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} \frac{|f_k^n(x_k)|}{n^r} \leq M\|x\|.
\]

This completes the proof.

**Theorem 3.2.** Let \( p = (p_k) \) be a bounded sequence of positive real numbers with \( p_k > 1 \) for all \( k \in \mathbb{N} \) and \( 1/p_k + 1/q_k = 1 \) for all \( k \in \mathbb{N} \), and let \( r \geq 0 \). For an infinite matrix \( A = (f_k^n) \), \( A \in (\ell(X,p),E_r) \) if and only if there is \( m_0 \in \mathbb{N} \) such that

\[
\sup_n \sum_{k=1}^{\infty} ||f_k^n||^{q_k} n^{-rq_k} m_0^{-q_k} < \infty.
\]

**Proof.** Let \( x = (x_k) \in \ell(X,p) \). By (3.6), there are \( m_0 \in \mathbb{N} \) and \( K > 1 \) such that

\[
\sum_{k=1}^{\infty} ||f_k^n||^{q_k} n^{-rq_k} m_0^{-q_k} < K, \quad \forall n \in \mathbb{N}.
\]

Note that for \( a, b \geq 0 \), we have

\[
ab \leq a^{p_k} + b^{q_k}.
\]
It follows by (3.7) and (3.8) that for \( n \in \mathbb{N} \),
\[
|n^{-r} \sum_{k=1}^{\infty} f^n_k(x_k)| = |n^{-r} \sum_{k=1}^{\infty} f^n_k(m_0^{-1} \cdot m_0 x_k)|
\]
\[
\leq \sum_{k=1}^{\infty} (n^{-r} m_0^{-1} ||f^n_k||) (||m_0 x_k||)
\]
\[
\leq \sum_{k=1}^{\infty} n^{-r q_k} m_0^{-d_k} ||f^n_k||^{q_k} + m_0^{\alpha} \sum_{k=1}^{\infty} ||x_k||^{p_k}
\]
\[
\leq K + m_0^{\alpha} \sum_{k=1}^{\infty} ||x_k||^{p_k}, \quad \text{where} \quad \alpha = \sup_k p_k.
\]

Hence \( \sup n^{-r} |\sum_{k=1}^{\infty} f^n_k(x_k)| < \infty \), so that \( Ax \in E_r \).

For necessity, assume that \( A \in (\ell(X, p), E_r) \). For each \( k \in \mathbb{N} \), we have \( \sup_n n^{-r} |f^n_k(x)| < \infty \) for all \( x \in X \) since \( e^{(k)}(x) \in \ell(X, p) \). It follows by the uniform bounded principle that for each \( k \in \mathbb{N} \) there is \( C_k > 1 \) such that
\[
\sup_n n^{-r} ||f^n_k|| \leq C_k.
\] 

(3.10)

Suppose that (3.6) is not true. Then
\[
\sup n^{-r} ||f^n_k|| = \infty, \quad \forall m \in \mathbb{N}.
\] 

(3.11)

For \( n \in \mathbb{N} \), we have by (3.10) that for \( k, m \in \mathbb{N} \),
\[
\sum_{j=1}^{\infty} ||f^n_j||^{q_j} n^{-r q_j} m^{-d_j} = \sum_{j=1}^{k} ||f^n_j||^{q_j} n^{-r q_j} m^{-d_j} + \sum_{j=k}^{\infty} ||f^n_j||^{q_j} n^{-r q_j} m^{-d_j}
\]
\[
\leq \sum_{j=1}^{k} C_j^{q_j} m^{-d_j} + \sum_{j=k}^{\infty} ||f^n_j||^{q_j} n^{-r q_j} m^{-d_j}.
\] 

(3.12)

This together with (3.11) give
\[
\sup n \sum_{j>k} ||f^n_j||^{q_j} n^{-r q_j} m^{-d_j} = \infty, \quad \forall k, m \in \mathbb{N}.
\] 

(3.13)

By (3.13) we can choose \( 0 = k_0 < k_1 < k_2 < \cdots, m_1 < m_2 < \cdots, m_i > 4^i \) and a subsequence \((n_i)\) of positive integers such that for all \( i \geq 1 \),
\[
\sum_{k_{i-1} < j \leq k_i} ||f_j^{n_i}||^{q_j} n_i^{-r q_j} m_i^{-d_j} > 2^i.
\] 

(3.14)

For each \( i \in \mathbb{N} \), we can choose \( x_j \in X \) with \( ||x_j|| = 1 \), for \( k_{i-1} < j \leq k_i \) such that
\[
\sum_{k_{i-1} < j \leq k_i} ||f_j^{n_i}(x_j)||^{q_j} n_i^{-r q_j} m_i^{-d_j} > 2^i.
\] 

(3.15)
For each $i \in \mathbb{N}$, let $F_i : (0, \infty) \to (0, \infty)$ be defined by

$$F_i(M) = \sum_{k_{i-1} < j \leq k_i} |f_{j}^{n_i}(x_j)|^{q_j} n_i^{-r q_j} M^{-q_j}.$$  \hspace{1cm} \text{(3.16)}

Then $F_i$ is continuous and non-increasing such that $F(M) \to 0$ as $M \to \infty$. Thus there exists $M_i > 0$ such that $M_i > m_i$ and

$$F(M_i) = \sum_{k_{i-1} < j \leq k_i} |f_{j}^{n_i}(x_j)|^{q_j} n_i^{-r q_j} M_i^{-q_j} = 2^i.$$ \hspace{1cm} \text{(3.17)}

Put

$$y = (y_j), \quad y_j = 4^{-i} M_i^{-(q_j-1)} n_i^{-r q_j/p_j} |f_{j}^{n_i}(x_j)|^{q_j-1} x_j \text{ for } k_{i-1} < j \leq k_i.$$ \hspace{1cm} \text{(3.18)}

Thus

$$\sum_{j=1}^{\infty} ||y_j||^{p_j} = \sum_{i=1}^{\infty} \sum_{k_{i-1} < j \leq k_i} 4^{-i p_j} M_i^{-p_j(q_j-1)} n_i^{-r q_j} |f_{j}^{n_i}(x_j)|^{p_j(q_j-1)}$$

$$\leq \sum_{i=1}^{\infty} 4^{-i} \sum_{k_{i-1} < j \leq k_i} M_i^{-q_j} n_i^{-r q_j} |f_{j}^{n_i}(x_j)|^{q_j}$$

$$= \sum_{i=1}^{\infty} 4^{-i} \cdot 2^i$$

$$= \sum_{i=1}^{\infty} \frac{1}{2^i} = 1.$$ \hspace{1cm} \text{(3.19)}

Thus $y = (y_j) \in \ell(X, p)$. Since $\ell(X, p)$ is a BK-space which is normal and norm monotone under the Luxemburg norm, by Lemma 3.1, we obtain that

$$\sup_n \sum_{k=1}^{\infty} |f_k^n(y_k)|^{n^r} < \infty.$$ \hspace{1cm} \text{(3.20)}

But we have

$$\sup_n \sum_{j=1}^{\infty} \frac{|f_j^n(y_j)|}{n^r} \geq \sup_i \sum_{j=1}^{\infty} \frac{|f_j^{n_i}(y_j)|}{n_i^r} \geq \sup_i \sum_{k_{i-1} < j \leq k_i} |f_{j}^{n_i}(x_j)|^{q_j}$$

$$= \sup_i \sum_{k_{i-1} < j \leq k_i} 4^{-i} M_i^{-(q_j-1)} n_i^{-r(q_j/p_j+1)} |f_{j}^{n_i}(x_j)|^{q_j}$$

$$= \sup_i \sum_{k_{i-1} < j \leq k_i} 4^{-i} M_i^{-(q_j-1)} n_i^{-r q_j} |f_{j}^{n_i}(x_j)|^{q_j}$$

$$= \sup_i \sum_{k_{i-1} < j \leq k_i} |f_{j}^{n_i}(x_j)|^{q_j} n_i^{-r q_j} M_i^{-q_j} 4^{-i} M_i$$

$$\geq \sup_i 2^i = \infty, \quad \text{because } M_i > 4^i.$$ \hspace{1cm} \text{(3.21)}

This is contradictory with (3.20). Therefore (3.6) is satisfied. \hfill \Box
**Theorem 3.3.** Let \( p = (p_k) \) be a bounded sequence of positive real numbers such that \( p_k > 1 \) for all \( k \in \mathbb{N} \), \( 1/p_k + 1/q_k = 1 \) for all \( k \in \mathbb{N} \), \( r \geq 0 \) and \( s \geq 0 \). Then for an infinite matrix \( A = (f^n_k), A \in (F_r(X,p),E_s) \) if and only if there is \( m_0 \in \mathbb{N} \) such that
\[
\sup_{n \to \infty} \sum_{k=1}^{\infty} \left( k^{-r/q_k/p_k} \| f^n_k \|^{q_k} n^{-s/q_k} m_0^{-q_k} \right) < \infty.
\] (3.22)

**Proof.** Since \( F_r(X,p) = \ell(X,p)(kr/p_k) \), it is easy to see that \( A \in (F_r(X,p),E_s) \iff (k^{-r/p_k} f^n_k)_{n,k} \in (\ell(X,p)E_s). \) (3.23)

By Theorem 3.2, we have \( (k^{-r/p_k} f^n_k)_{n,k} \in (\ell(X,p)E_s) \) if and only if there is \( m_0 \in \mathbb{N} \) such that
\[
\sup_{n \to \infty} \sum_{k=1}^{\infty} \left( k^{-r/q_k/p_k} \| f^n_k \|^{q_k} n^{-s/q_k} m_0^{-q_k} \right) < \infty.
\] (3.24)

Thus the theorem is proved. \( \square \)

Since \( E_0 = \ell_\infty \), the following two results are obtained directly from Theorems 3.2 and 3.3, respectively.

**Corollary 3.4.** Let \( p = (p_k) \) be a bounded sequence of positive real numbers with \( p_k > 1 \) for all \( k \in \mathbb{N} \) and let \( 1/p_k + 1/q_k = 1 \) for all \( k \in \mathbb{N} \). Then for an infinite matrix \( A = (f^n_k), A \in (\ell(X,p),\ell_\infty) \) if and only if there is \( m_0 \in \mathbb{N} \) such that
\[
\sup_{n \to \infty} \sum_{k=1}^{\infty} \| f^n_k \|^{q_k} m_0^{-q_k} < \infty.
\] (3.25)

**Corollary 3.5.** Let \( p = (p_k) \) be a bounded sequence of positive real numbers with \( p_k > 1 \) for all \( k \in \mathbb{N} \) and let \( 1/p_k + 1/q_k = 1 \) for all \( k \in \mathbb{N} \). Then for an infinite matrix \( A = (f^n_k), A \in (F_r(X,p),\ell_\infty) \) if and only if there is \( m_0 \in \mathbb{N} \) such that
\[
\sup_{n \to \infty} \sum_{k=1}^{\infty} \left( k^{-r/q_k/p_k} \| f^n_k \|^{q_k} n^{-s/q_k} m_0^{-q_k} \right) < \infty.
\] (3.26)

**Theorem 3.6.** Let \( p = (p_k) \) and \( q = (q_k) \) be bounded sequences of positive real numbers with \( p_k > 1 \) for all \( k \in \mathbb{N} \) and let \( 1/p_k + 1/t_k = 1 \) for all \( k \in \mathbb{N} \). Then for an infinite matrix \( A = (f^n_k), A \in (\ell(X,p),\ell_\infty(q)) \) if and only if for each \( r \in \mathbb{N} \), there is \( m_r \in \mathbb{N} \) such that
\[
\sup_{n,k} \sum_{k=1}^{\infty} r^{-t_k/q_n} \| f^n_k \|^{t_k} m_r^{-t_k} < \infty.
\] (3.27)

**Proof.** Since \( \ell_\infty(q) = \cap_{r=1}^{\infty} \ell_\infty(r^{1/q_k}) \), it follows that
\[
A \in (\ell(X,p),\ell_\infty(q)) \iff A \in \left( \ell(X,p),\ell_\infty(r^{1/q_k}) \right), \quad \forall r \in \mathbb{N}.
\] (3.28)

It is easy to show that for \( r \in \mathbb{N} \),
\[
A \in \left( \ell(X,p),\ell_\infty(r^{1/q_k}) \right) \iff (r^{1/q_n} f^n_k)_{n,k} \in (\ell(X,p),\ell_\infty).
\] (3.29)
We obtain by Corollary 3.4 that for \( r \in \mathbb{N}, (r^{1/q_n} f^n_{k,n,k})_{n,k} \in (\ell(X,p), \ell_\infty) \) if and only if there is \( m_r \in \mathbb{N} \) such that

\[
\sup_n \sum_{k=1}^{\infty} r^{t_k/q_n} \|f^n_k\|^{t_k} m_r^{-t_k} < \infty. 
\] (3.30)

Thus the theorem is proved.

**Theorem 3.7.** Let \( p = (p_k) \) and \( q = (q_k) \) be bounded sequences of positive real numbers with \( p_k > 1 \) for all \( k \in \mathbb{N} \) and let \( 1/p_k + 1/t_k = 1 \) for all \( k \in \mathbb{N} \). For an infinite matrix \( A = (f^n_k), A \in (F_r(X,p), \ell_\infty(q)) \) if and only if for each \( i \in \mathbb{N} \), there is \( m_i \in \mathbb{N} \) such that

\[
\sup_n \sum_{k=1}^{\infty} i^{t_k/q_n} k^{1-rt_k/p_k} \|f^n_k\|^{t_k} m_i^{-t_k} < \infty.
\] (3.31)

**Proof.** Since \( F_r(X,p) = \ell(X,p)(k^{r/p_k}), \) it implies that

\[
A \in (F_r(X,p), \ell_\infty(q)) \iff (k^{-r/p_k} f^n_{k,n,k})_{n,k} \in (\ell(X,p), \ell_\infty(q)).
\] (3.32)

It follows from Theorem 3.6 that \( A \in (F_r(X,p), \ell_\infty(q)) \) if and only if for each \( i \in \mathbb{N}, \) there is \( m_i \in \mathbb{N} \) such that

\[
\sup_n \sum_{k=1}^{\infty} i^{t_k/q_n} k^{1-rt_k/p_k} \|f^n_k\|^{t_k} m_i^{-t_k} < \infty.
\] (3.33)

**Theorem 3.8.** Let \( p = (p_k) \) be bounded sequence of positive real numbers with \( p_k > 1 \) for all \( n \in \mathbb{N} \) and let \( 1/p_k + 1/q_k = 1 \) for all \( k \in \mathbb{N} \). Then for an infinite matrix \( A = (f^n_k), A \in (\ell(X,p), bs) \) if and only if there is \( m_0 \in \mathbb{N} \) such that

\[
\sup_n \sum_{k=1}^{\infty} \left( \sum_{i=1}^{n} f^i_k \right)^{q_k} m_0^{-q_k} < \infty.
\] (3.34)

**Proof.** For an infinite matrix \( A = (f^n_k), \) we can easily show that

\[
A \in (\ell(X,p), bs) \iff \left( \sum_{i=1}^{n} f^i_k \right)_{n,k} \in (\ell(X,p), \ell_\infty).
\] (3.35)

This implies by Corollary 3.4 that \( A \in (\ell(X,p), bs) \) if and only if there is \( m_0 \in \mathbb{N} \) such that

\[
\sup_n \sum_{k=1}^{\infty} \left( \sum_{i=1}^{n} f^i_k \right)^{q_k} m_0^{-q_k} < \infty.
\] (3.36)

**Theorem 3.9.** Let \( p = (p_k) \) be a bounded sequence of positive real numbers with \( p_k > 1 \) for all \( k \in \mathbb{N} \) and let \( 1/p_k + 1/q_k = 1 \) for all \( k \in \mathbb{N} \). Then for an infinite matrix \( A = (f^n_k), A \in (\ell(X,p), cs) \) if and only if

1. there is \( m_0 \in \mathbb{N} \) such that \( \sup_n \sum_{k=1}^{\infty} \|f^n_k\|^{q_k} m_0^{-q_k} < \infty \) and
2. for each \( k \in \mathbb{N} \) and \( x \in X, \sum_{n=1}^{\infty} f^n_k(x) \) converges.
**Proof.** The necessity is obtained by Theorem 3.8 and by the fact that $e^{(k)}(x) \in \ell(X,p)$ for every $k \in \mathbb{N}$ and $x \in X$.

Now, suppose that (1) and (2) hold. By Theorem 3.8, we have $A : \ell(X,p) \rightarrow bs$. Let $x = (x_k) \in \ell(X,p)$. Since $\ell(X,p)$ has the AK property, we have $x = \lim_{n \rightarrow \infty} \sum_{k=1}^{n} e^{(k)}(x_k)$.

By Zeller’s theorem, $A : \ell(X,p) \rightarrow bs$ is continuous. It implies that

$$Ax = \lim_{n \rightarrow \infty} \sum_{k=1}^{n} Ae^{(k)}(x_k).$$

(3.37)

By (2), $Ae^{(k)}(x_k) \in cs$ for all $k \in \mathbb{N}$. Since $cs$ is a closed subspace of $bs$, it implies that $Ax \in cs$, that is, $A : \ell(X,p) \rightarrow cs$.

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**References**


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