ON MINIMAL ARTINIAN MODULES AND MINIMAL ARTINIAN LINEAR GROUPS

LEONID A. KURDACHENKO and IGOR YA. SUBBOTIN

(Received 19 March 2001)

ABSTRACT. The paper is devoted to the study of some important types of minimal artinian linear groups. The authors prove that in such classes of groups as hypercentral groups (so also, nilpotent and abelian groups) and $\mathcal{FC}$-groups, minimal artinian linear groups have precisely the same structure as the corresponding irreducible linear groups.

2000 Mathematics Subject Classification. 20E36, 20F28.

Let $F$ be a field, $A$ a vector space over $F$. The group $\text{GL}(F,A)$ of all automorphisms of $A$ and its distinct subgroups are the oldest subjects of investigation in Group Theory. For the case when $A$ has a finite dimension over $F$, every element of $\text{GL}(F,A)$ defines some nonsingular $n \times n$-matrix over $F$, where $n = \dim_F A$. Thus, for the finite-dimensional case, the theory of linear groups is exactly the theory of matrix groups. That is why the theory of finite-dimensional linear groups is one of the best developed in algebra. However, for the case when $\dim_F A$ is infinite, the situation is totally different. The study of this case always requires some essential additional restrictions. Thus, the transition from the study of finite groups to the study of infinite groups generated the finiteness conditions. It is natural to apply these finiteness conditions to the study of infinite-dimensional linear groups. The study of finitary linear groups (the linear analogies of $\mathcal{FC}$-groups) shows the effectiveness of such approach (cf. a survey of Phillips [6]).

The groups having a finite composition series were one of the first generalization of the finite groups. Let $G \leq \text{GL}(F,A)$, then we can consider $A$ as an $FG$-module. We say that $A$ has a finite composition length if $A$ has a finite series $\langle 0 \rangle = B_0 \leq B_1 \leq \cdots \leq B_n = A$ of $FG$-submodules, every factor of which is a simple $FG$-module. We can consider $G/C_G(B_{i+1}/B_i)$ as an irreducible linear group, $0 \leq i \leq n - 1$. Let $T = \bigcap_{0 \leq i \leq n - 1} C_G(B_{i+1}/B_i)$; then $G/T \cong \chi_{0 \leq i \leq n - 1} G/C_G(B_{i+1}/B_i)$, and $T$ is a nilpotent bounded $p$-subgroup whenever $\text{char} F = p$, or $T$ is a nilpotent divisible torsion-free subgroup whenever $\text{char} F = 0$. Thus, the case of irreducible linear groups is basic. Irreducible linear groups as the automorphism groups of abelian chief factors, play a crucial role in Group Theory, and their investigation is very useful for the solution of many group theoretical problems. For the infinite-dimensional case, the irreducible groups under some natural restrictions have been studied by Hartley and McDougall [2], Zaicov [13], Robinson and Zhang [9], Franciosi, de Giovanni, and Kurdachenko [1], and Kurdachenko and Subbotin [5].
The minimal and the maximal conditions were the very next classical finiteness conditions that have appeared in algebra. Note that every $FG$-module of finite composition length is artinian (i.e., it satisfies the minimal condition on $FG$-submodules) and noetherian (i.e., it satisfies the maximal condition on $FG$-submodules).

Let $R$ be a ring, $A$ an artinian $R$-module. Put

$$S_{\text{nd}}(A) = \{ B \mid B \text{ is an } R\text{-submodule of } A \text{ and has no finite composition series} \}. \tag{1}$$

If $A$ has no finite composition series, then $S_{\text{nd}}(A) \neq \emptyset$. Since $A$ is artinian, $S_{\text{nd}}(A)$ has a minimal element $M$. Thus, if $U$ is a proper $R$-submodule of $M$, then $U$ has a finite composition length.

An $R$-module $M$ is said to be a minimal artinian, if $M$ has no finite composition series, but each of its proper submodule has a finite composition length.

Thus every artinian module includes a minimal artinian submodule. On the other hand, the structure of artinian modules depends on the structure of its minimal artinian submodules, therefore, the study of minimal artinian modules is one of the important steps for the study of artinian modules.

Let again $F$ be a field, $A$ a vector space over $F$, $G \leq \text{GL}(F, A)$. We want to consider the situation when $A$ is a minimal artinian $FG$-module. This consideration will lead us to the fact that the group $G$ is lying in the class $\mathcal{X}$ such that all irreducible $\mathcal{X}$-groups have been described. So we may set that if an $FG$-module $B$ has finite composition series, then the structure of $G$ is defined.

Let $F$ be a field, $A$ a vector space over $F$, $G \leq \text{GL}(A)$. A group $G$ is called a minimal artinian if the following conditions hold:

1. $A$ has no finite composition series;
2. if $B$ is a proper $FG$-submodule of $A$, then $B$ has a finite composition length.

The study of minimal artinian $FG$-modules (as any $FG$-module) consists of two parts: the study of internal structure of the module and the study of the group $G/C_G(A)$. The last group is imbedded in $\text{GL}(F, A)$, that is, it is a linear minimal artinian group. Our paper is devoted to the study of some important types of minimal artinian linear groups. The main results of this paper show that in such classes of groups as hypercentral groups (so also, nilpotent and abelian groups) or $FC$-groups the minimal artinian linear groups have precisely the same structure as the corresponding irreducible linear groups have.

Now we mention some needed results on hypercentral irreducible groups. The irreducible $ZG$-modules have been studied in [5]. These results can be extended almost without changes on the case of irreducible subgroups of $\text{GL}(F, A)$, where $A$ is a vector space over a field $F$.

**Lemma 1.** Let $F$ be a field, $G$ a group, $A$ a simple $FG$-module, $I = \text{Ann}_{FG}(A)$. If $C/I$ is a center of $FG/I$, then $C/I$ is an integral domain. In particular, the periodic part of $\zeta(G/C_G(A))$ is a locally cyclic $p'$-subgroup where $p = \text{char } F$.

As usual, $0'$ denotes the set of all primes.

This statement is an immediate corollary of the known theorem of I. Schur.

A group $G$ is said to have finite 0-rank $r_0(G) = r$ (or finite torsion-free rank) if $G$ has a finite subnormal series with exactly $r$ infinite cyclic factors being the others periodic.
We note that every refinement of each of these series has only \( r \) infinite cyclic factors. Since every two subnormal series have the isomorphic refinements, 0-rank is independent of the choice of the subnormal series.

Note also that if \( G \) is a locally nilpotent group of finite 0-rank, then the factor-group \( G/t(G) \) by the periodic part \( t(G) \) has a finite special rank.

**Lemma 2.** Let \( G \) be a hypercentral group of finite 0-rank, \( F \) a locally finite field, \( A \) a simple \( FG \)-module. Then \( \zeta(G/C_G(A)) \) is periodic.

This lemma follows from [4, Theorem 2].

**Lemma 3.** Let \( F \) be a field, \( p = \text{char} F \), \( G \) an abelian group of finite 0-rank.

(1) If the field \( F \) is locally finite, and \( G \) is a locally cyclic \( p' \)-group, then there exists a simple \( FG \)-module \( A \) such that \( C_G(A) = \langle 1 \rangle \).

(2) If \( F \) is not locally finite, and \( t(G) \) is a locally cyclic \( p' \)-group, then there exists a simple \( FG \)-module \( A \) such that \( C_G(A) = \langle 1 \rangle \).

This construction is contained in [2].

**Lemma 4.** Let \( F \) be a field, \( p = \text{char} F \), \( G \) an abelian group of infinite 0-rank. If \( t(G) \) is a locally cyclic \( p' \)-group, then there exists a simple \( FG \)-module \( A \) such that \( C_G(A) = \langle 1 \rangle \).

This assertion has been proved in [9] for the case of finite field, however it is valid also for an arbitrary field.

**Lemma 5.** Let \( F \) be a field, \( p = \text{char} F \), \( G \) a hypercentral group of finite 0-rank, \( C = \zeta(G) \), \( T = t(C) \).

(1) If the field \( F \) is locally finite, and \( C = T \) is a locally cyclic \( p' \)-group, then there exists a simple \( FG \)-module \( A \) such that \( C_G(A) = \langle 1 \rangle \).

(2) If \( F \) is not locally finite and \( T \) is a locally cyclic \( p' \)-group, then there exists a simple \( FG \)-module \( A \) such that \( C_G(A) = \langle 1 \rangle \).

**Lemma 6.** Let \( F \) be a field, \( p = \text{char} F \), \( G \) a hypercentral group of infinite 0-rank, \( C = \zeta(G) \), \( T = t(C) \). If \( T \) is a locally cyclic \( p' \)-group, then there exists a simple \( FG \)-module \( A \) such that \( C_G(A) = \langle 1 \rangle \).

The proof of both these assertions is similar to the proof of the respective results of [5].

**Lemma 7.** Let \( R \) be a ring, \( A \) a minimal artinian \( R \)-module. Then \( A \) does not decompose into a direct sum of two proper \( R \)-submodules.

The lemma is obvious.

If \( A \) is an \( R \)-module, then let \( \text{Soc}_R(A) \) denotes the sum of all minimal \( R \)-submodules whenever \( A \) includes such submodules, and \( \text{Soc}_R(A) = \langle 0 \rangle \) otherwise.

Clearly, \( \text{Soc}_R(A) \) is a direct sum of some minimal \( R \)-submodules (if it is nonzero). If \( A \) is an artinian \( R \)-module, then \( \text{Soc}_R(A) \neq \langle 0 \rangle \) and \( \text{Soc}_R(A) \) is a direct sum of finitely many minimal \( R \)-submodules. So we come to the following lemma.

**Lemma 8.** Let \( R \) be a ring, \( A \) a minimal artinian \( R \)-module. Then \( \text{Soc}_R(A) \) is a nonzero proper submodule of \( A \).
**Lemma 9.** Let $F$ be a field, $G$ a group, $H$ a normal subgroup having a finite index in $G$, $A$ an $FG$-module. If $A$ has finite composition length as an $FG$-module, then $A$ has finite composition length as an $FH$-module.

**Proof.** Let
\[ \langle 0 \rangle = B_0 \leq B_1 \leq \cdots \leq B_n = A \quad (2) \]
be a series of $FG$-submodules with simple $FG$-factors. Then $B_{i+1}/B_i$ is a direct sum of finitely many simple $FH$-submodules \[12\], $0 \leq i \leq n-1$. Thus $A$ has a finite series of $FH$-submodules with simple factors.

**Proposition 10.** Let $F$ be a field, $G$ a group, $A$ a minimal artinian $FG$-module such that $C_G(A) = \langle 1 \rangle$, $H$ a normal subgroup having in $G$ finite index, $X$ a transversal to $H$ in $G$. Then

1. $A$ includes a minimal artinian $FH$-submodule $B$;
2. $A = \sum_{x \in X} Bx$;
3. $\bigcap_{x \in X} x^{-1}C_H(B)x = \langle 1 \rangle$, in particular, $H \leq x_{x \in X} H/(x^{-1}C_H(B)x)$.

**Proof.** By Wilson’s theorem \[11\] $A$ is an artinian $FH$-module. Since $A$ has no finite composition series as $FG$-module, then $A$ has no finite composition series as $FH$-module by Lemma 9. Let
\[ \mathcal{S} = \{ U \mid U \text{ is an } FH\text{-submodule of } A \text{ and has no finite composition series} \}. \quad (3) \]

Since $A \in \mathcal{S}$, $\mathcal{S} \neq \emptyset$. Then $\mathcal{S}$ has a minimal element $B$. This means that $B$ is minimal artinian $FH$-submodule. The sum $C = \sum_{x \in X} Bx$ is an $FG$-submodule. If we suppose that $C$ is a proper $FG$-submodule of $A$, then it has a finite composition length. By Lemma 9 it has also a finite composition length as an $FH$-module, which contradicts the choice of $B$. This contradiction proves the equality $A = \sum_{x \in X} Bx$. Since $C_H(Bx) = x^{-1}C_H(B)x$, then it follows that $\bigcap_{x \in X} x^{-1}C_H(B)x \leq C_H(A) = \langle 1 \rangle$. By Remak’s theorem, $H \leq x_{x \in X} H/(x^{-1}C_H(B)x)$.

**Lemma 11.** Let $F$ be a field, $G$ a group, $A$ a minimal artinian $FG$-module such that $C_G(A) = \langle 1 \rangle$. If $1 \neq x \in \zeta(G)$, then $A = A(x-1)$.

**Proof.** The mapping $\varphi : a \rightarrow a(x-1)$, $a \in A$, is an $FG$-endomorphism of $A$. In particular, $\text{Im}\varphi = A(x-1)$ and $\text{Ker}\varphi = C_A(x)$ are the $FG$-submodules of $A$. Since $x \in C_G(A)$, then $C_A(x) \neq A$. By $A(x-1) \cong A/C_A(x)$, we obtain that $A(x-1)$ has no finite composition length. It follows that $A(x-1) = A$.

**Corollary 12.** Let $F$ be a field, $A$ a vector space over $F$, $G$ a minimal artinian subgroup of $GL(F,A)$. Suppose that $G$ is hypercentral. If $\text{char} F = p > 0$, then $G$ does not contain $p$-elements.

**Proof.** Denote by $P$ the Sylow $p$-subgroup of $G$, and suppose that $P \neq \langle 1 \rangle$. Since $G$ is a hypercentral group, $P \cap \zeta(G) \neq \langle 1 \rangle$. Let $1 \neq z \in \zeta(G) \cap P$. Since the additive group of $A$ is an elementary abelian $p$-group, a natural semidirect product $B \times \langle z \rangle$ is a nilpotent group (cf. \[8, Lemma 6.34\]). Therefore $[A(z), A(z)] = A(z-1) \neq A$, which contradicts Lemma 11. This contradiction shows that $P = \langle 1 \rangle$. \[\square\]
Let $G$ be a group. Put
\[ FC(G) = \{ x \in G \mid x^G = \{ g^{-1}xg \mid g \in G \} \text{ is finite} \}. \tag{4} \]

That is, $FC(G)$ is a characteristic subgroup of $G$. This subgroup is called the $FC$-center of $G$.

Furthermore, the set $T$ of all elements of finite order is a (characteristic) subgroup of $FC(G)$ and $FC(G)/T$ is an abelian torsion-free group (cf. [7, Theorem 4.32]).

Let $G$ be a group, $\pi$ a set of primes. Denote by $O_{\pi}(G)$ the maximal normal $\pi$-subgroup of $G$. In particular, if $p$ is prime, then $O_p(G)$ denotes the maximal normal $p$-subgroup of $G$, and $O'_p(G)$ denotes the maximal periodic subgroup, which does not contain the $p$-elements.

**Corollary 13.** Let $F$ be a field, $A$ a vector space over $F$, $G$ a minimal artinian subgroup of $GL(F,A)$. If $char F = p > 0$, then $O_p(FC(G)) = \{1\}$.

**Proof.** Suppose the contrary, let $1 \neq y \in O_p(FC(G))$. Put $Y = \langle y \rangle^G$. By Ditsmann’s lemma (cf. [7, Corollary 2 to Lemma 2.14]), $Y$ is a finite normal subgroup of $G$. Since $Y$ is a finite $p$-subgroup, $\zeta(Y) = Z \neq \{1\}$. Let $H = CG(Z)$, then $H$ is a normal subgroup of finite index, and $Z \leq \zeta(H)$. By Proposition 10 $A$ includes a minimal artinian $FH$-submodule $B$. Since the additive group of $B$ is an elementary abelian $p$-group, the natural semidirect product $B \times \langle z \rangle$ is a nilpotent group for each element $z \in Z$ (cf. [8, Lemma 6.34]). Therefore $[B(z), B(z)] = B(z-1) \neq B$. Corollary 12 yields that $z \in CG(B)$. It is valid for every element $z \in Z$, therefore $Z \leq CG(B)$. In turn $Z = x^{-1}zx \leq x^{-1}CG(B)x = CG(Bx)$ for an arbitrary element $x \in G$. Since it is true for every element $x \in G$, $Z \leq \bigcap_{x \in \Lambda} CG(Bx) = CG(A)$, because $A = \sum_{x \in \Lambda} Bx$. But $CG(A) = \langle 1 \rangle$. This contradiction proves that $O_p(FC(G)) = \{1\}$. \qed

**Corollary 14.** Let $F$ be a field, $A$ a vector space over $F$, $G$ a minimal artinian subgroup of $GL(F,A)$. If $char F = p > 0$, then the locally soluble radical of $FC(G)$ has no $p$-elements.

**Lemma 15.** Let $F$ be a field, $char F = p$, $A$ a vector space over $F$, $G$ a minimal artinian subgroup of $GL(F,A)$. If $H$ is a nonidentity finite normal $p'$-subgroup of $G$, then $\text{Soc}_{FH}(A) = A$.

**Proof.** For every element $0 \neq a \in A$, an $FH$-submodule $aFH$ is finite-dimensional. In particular, it includes a simple $FH$-submodule. This means that $\text{Soc}_{FH}(A) \neq \{0\}$. By Maschke’s theorem (cf. [10, Theorem 1.5]), $\text{Soc}_{FH}(A) = A$. \qed

If $R$ is a ring, $G$ a group, then $\omega RG$ denotes the augmentation ideal of the group ring $RG$.

**Corollary 16.** Let $F$ be a field, $char F = p$, $A$ a vector space over $F$, $G$ a minimal artinian subgroup of $GL(F,A)$. If $H$ is a nonidentity finite normal $p'$-subgroup of $G$, then $C_A(H) = \{0\}$, $A(\omega FH) = A$.

**Proof.** By Lemma 15, $A = \bigoplus_{\lambda \in \Lambda} M_\lambda$, where $M_\lambda$ is a simple $FH$-submodule, $\lambda \in \Lambda$. Since $M_\lambda(\omega FH)$ is an $FH$-submodule of $M_\lambda$, then either $M_\lambda(\omega FH) = M_\lambda$ or $M_\lambda(\omega FH) = \{0\}$. It implies the equality $A = C_A(H) \oplus A(\omega FH)$. Since $H$ is a normal subgroup of
$G$, both $C_A(H)$ and $A(\omega FH)$ are $FG$-submodules. Lemma 7 yields that $C_A(H) = (0)$ and $A = A(\omega FH)$.

**Corollary 17.** Let $F$ be a field, char $F = p$, $A$ a vector space over $F$, $G$ a minimal artinian subgroup of $GL(F, A)$. If $H$ is a nonidentity finite normal $p'$-subgroup of $G$ and $B$ is a nonzero $FG$-submodule of $A$, then $C_H(B) = (1)$.

**Proof.** In fact, if $H_1 = C_H(B) \neq (1)$, then $H_1$ is a nonidentity finite normal $p'$-subgroup of $G$. Since $B \leq C_A(H_1)$, we obtain a contradiction with Corollary 12.

**Corollary 18.** Let $F$ be a field, char $F = p$, $A$ a vector space over $F$, $G$ a minimal artinian subgroup of $GL(F, A)$. Furthermore, let $H$ be a nonidentity normal $p'$-subgroup having an ascending series of $G$-invariant subgroups

$$(1) = H_0 \leq H_1 \leq \cdots \leq H_{\alpha} \leq H_{\alpha+1} \leq \cdots \leq H_\gamma = H$$

with finite factors. If $B$ is a nonzero proper $FG$-submodule of $A$, then $C_H(B) = (1)$.

**Proof.** We use induction on $\alpha$. If $\alpha = 1$, then the assertion follows from Corollary 13. Let $\alpha > 1$, and we have already proved that $C_{H_{\beta'}}(B) = (1)$ for all $\beta < \alpha$.

Let $C_{\alpha} = C_{H_{\alpha}}(B)$. If $\alpha$ is a limit ordinal, then $H_{\alpha} = \bigcup_{\beta<\alpha} H_{\beta}$, and therefore $C_{\alpha} = \bigcup_{\beta<\alpha} (C_{\alpha} \cap H_{\beta})$. But $C_{\alpha} \cap H_{\beta} = C_{H_{\beta'}}(B) = (1)$. Thus $C_{\alpha} = (1)$.

Suppose now that $\alpha$ is not a limit, and put $L = H_{\alpha-1}$. Assume that $C_{\alpha} \neq (1)$. Then $C_{\alpha} \cap L = C_L(B) = (1)$, so that $C_{\alpha} \equiv C_{\alpha}/(C_{\alpha} \cap L) \equiv C_{\alpha}L/L \leq H_{\alpha}/L$. It follows that $C_{\alpha}$ is a finite normal subgroup of $G$. And we obtain a contradiction with Corollary 12 because $B \leq C_A(C_{\alpha})$. Hence $C_{H_{\alpha}}(B) = (1)$. For $\alpha = \gamma$ we obtain that $C_H(B) = (1)$.

Let $G$ be a group. A normal subgroup $H$ is called the hyperfinite radical of $G$ if $H$ satisfies the following conditions:

(1) $H$ possesses an ascending series of $G$-invariant subgroups

$$(1) = H_0 \leq H_1 \leq \cdots \leq H_{\alpha} \leq H_{\alpha+1} \leq \cdots \leq H_\gamma = H,$$

every factor of which is finite;

(2) $G/H$ has no nonidentity finite normal subgroups.

We will denote the hyperfinite radical of $G$ by $HF(G)$.

Let $\text{Soc}(G) = X_{\lambda \in \Lambda} S_{\lambda}$, where $S_{\lambda}$ is a minimal normal subgroup of $G$, $\lambda \in \Lambda$. Put

$$\Lambda_{ab} = \{ \lambda \in \Lambda \mid S_{\lambda} \text{ is abelian} \}, \quad \text{Soc}_{ab}(G) = X_{\lambda \in \Lambda_{ab}} S_{\lambda}. \quad (7)$$

**Corollary 19.** Let $F$ be a field, char $F = p$, $A$ a vector space over $F$, $G$ a minimal artinian subgroup of $GL(F, A)$. Let $S = \text{Soc}_{ab}(G) \cap HF(G)$. Then $S$ is a $p'$-subgroup including a subgroup $Q$ such that $S/Q$ is a locally cyclic group and $\text{Core}_G(Q) = (1)$.

**Proof.** Clearly $S$ is a subgroup of the locally soluble radical of $FC(G)$. By Corollary 17 of Lemma 11, $S$ is a $p'$-subgroup. Let $B$ be a minimal $FG$-submodule of $A$. By Corollary 18, $C_{S}(B) = (1)$. In other words, $S$ is imbedded in an irreducible subgroup of $GL(F, B)$. And now we can use [1, Lemma 8.2].

Now we can expose the main results.
**Theorem 20.** Let $F$ be a field, $A$ a vector space over $F$, $G$ a minimal Artinian subgroup of $\text{GL}(F,A)$. If $G$ is an $FC$-group, then $\text{Soc}_{ab}(G)$ is a $p'$-subgroup including a subgroup $Q$ such that $\text{Soc}_{ab}(G)/Q$ is a locally cyclic group and $\text{Core}_G(Q) = \langle 1 \rangle$, where $p = \text{char } F$.

**Proof.** Let $T$ be the periodic part of $G$, $S$ the locally soluble radical of $G$. For every element $x \in T$, the subgroup $\langle x \rangle^G$ is finite by Ditsmann’s lemma (cf. [7, Corollary 2 to Lemma 2.14]). This implies the inclusion $T \leq HF(G)$. In particular, $\text{Soc}_{ab}(G) \leq HF(G)$.

Now we can use Corollary 19 of Lemma 15. The results of [3] imply that for the group $G$ having the structure, described in Theorem 20, there is a simple $FG$-module $A$ such that $C_G(A) = \langle 1 \rangle$. This means that this theorem cannot be strengthened. Thus, minimal Artinian linear $FC$-groups have the same structure as irreducible linear $FC$-groups.

**Theorem 21.** Let $F$ be a field, $A$ a vector space over $F$, $G$ a minimal Artinian subgroup of $\text{GL}(F,A)$. If $G$ is a hypercentral, then $t(\zeta(G))$ is a locally cyclic $p'$-subgroup, where $p = \text{char } F$.

**Proof.** By Corollary 12 of Lemma 11, the periodic part $T$ of the group $G$ is a $p'$-subgroup. Since $G$ is a hypercentral group, $T = HF(G)$. Choose a minimal $FG$-submodule $B$ of $A$. By Corollary 18 of Lemma 15, $T \cap C_G(B) = \langle 1 \rangle$, that is, $T \cong TC_G(B) / C_G(B)$. In other words, $T$ is imbedded in an irreducible subgroup of $\text{GL}(F,B)$. Now we can use Lemma 1.

**Corollary 22.** Let $F$ be a field, $A$ a vector space over $F$, $G$ a minimal Artinian subgroup of $\text{GL}(F,A)$. If $G$ is abelian, then $t(G)$ is a locally cyclic $p'$-subgroup, where $p = \text{char } F$.

Lemmas 3, 4, 5, and 6 show that, for the group $G$ having the structure described in Theorem 21 (and in its corollary), there is a simple $FG$-module $A$ such that $C_G(A) = \langle 1 \rangle$. This means that this theorem (and its corollary) cannot be strengthened. Thus, minimal Artinian linear hypercentral (and abelian) groups have the same structure as irreducible linear hypercentral (abelian) groups.

In connection with Lemma 2 and Theorem 21, there arises the following question: let $F$ be a locally finite field, $G$ a hypercentral group of finite 0-rank. Let $G$ be a minimal Artinian subgroup of $\text{GL}(F,A)$. Can we claim $\zeta(G)$ to be periodic? The following simple example gives a negative answer to it.

Let $F$ be a field, $A$ a vector space over $F$ of countable dimension, $\{a_n \mid n \in \mathbb{N}\}$ a basis of $A$, $\langle x \rangle$ an infinite cyclic group. Define the action of $x$ on $A$ by the rule

$$a_1x = a_1, \quad a_{n+1}x = a_{n+1} + a_n, \quad \text{or} \quad a_1(x-1) = 0, \quad a_{n+1}(x-1) = a_n, \quad n \in \mathbb{N}. \quad (8)$$

Then we can consider $A$ as an $F(x)$-module. It is easy to see that $A = A(x-1)$ and every proper $F(x)$-submodule of $A$ coincides with some $a_1F + \cdots + a_nF$, $n \in \mathbb{N}$. In particular, the $F(x)$-module $A$ is minimal Artinian and $C_{(x)}(A) = \langle 1 \rangle$.

Also it shows that the question about the internal structure of minimal Artinian modules requires separate consideration.
References


Leonid A. Kurdachenko: Mathematics Department, Dnepropetrovsk University, Prokhorovka, 49050 Dnepropetrovsk, Ukraine
E-mail address: mmf@ff.dsu.dp.ua

Igor Ya. Subbotin: Mathematics Department, National University, 9920 S. La Cienega Blvd, Inglewood, CA 90301, USA
E-mail address: isubboti@nu.edu
Submit your manuscripts at http://www.hindawi.com