A NOTE ON MINIMAL ENVELOPES OF DOUGLAS ALGEBRAS, MINIMAL SUPPORT SETS, AND RESTRICTED DOUGLAS ALGEBRAS

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ABSTRACT. We characterize the interpolating Blaschke products of finite type in terms of their support sets. We also give a sufficient condition on the restricted Douglas algebra of a support set that is invariant under the Bourgain map, and its minimal envelope is singly generated.

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1. Introduction. Let H^∞ be the Banach algebra of bounded analytic functions on the open unit disk D . We denote by $M(H^\infty)$ the set of nonzero complex valued homomorphism of H^∞ . With the weak*-topology, $M(H^\infty)$ is a compact Hausdorff space. We identify a function in H^∞ with the Gelfand transform and consider H^∞ the supremum norm closed subalgebra of the space of continuous functions on $M(H^\infty)$. By Carleson's corona theorem, D is dense in $M(H^\infty)$ in the weak*-topology. For $f \in H^\infty$, put

$$\begin{aligned} Z(f) &= \{x \in M(H^\infty) \setminus D : f(x) = 0\}, \\ \{|f| < 1\} &= \{x \in M(H^\infty) \setminus D : |f(x)| < 1\}. \end{aligned} \quad (1.1)$$

For two points x, y in $M(H^\infty)$, the pseudohyperbolic distance is given by

$$\rho(x, y) = \sup \{|f(y)| : f \in H^\infty, \|f\|_\infty \leq 1, f(x) = 0\}. \quad (1.2)$$

Then, $0 \leq \rho(x, y) \leq 1$ and put

$$P(x) = \{m \in M(H^\infty) : \rho(x, m) < 1\}. \quad (1.3)$$

The set $P(x)$ is called the Gleason part containing x . For $z, w \in D$, $\rho(z, w) = |(z - w)/(1 - \bar{w}z)|$, and $P(z) = D$. When $P(x) \neq \{x\}$, both x and $P(x)$ are called nontrivial. We denote by G the set of nontrivial points in $M(H^\infty)$.

For an infinite sequence $\{z_n\}_n$ in D with $\sum_{n=1}^\infty (1 - |z_n|) < \infty$, the corresponding Blaschke product is defined by

$$b(z) = \prod_{n=1}^\infty \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}, \quad z \in D. \quad (1.4)$$

In addition, we have

$$\inf_n (1 - |z_n|^2) |b'(z_n)| > 0, \quad (1.5)$$

both b and $\{z_n\}_n$ are called interpolating. When b is interpolating and

$$\lim_{n \rightarrow \infty} (1 - |z_n|^2) |b'(z_n)| = 1, \quad (1.6)$$

both b and $\{z_n\}_n$ are called sparse. An interpolating Blaschke product b is said to be unimodular on trivial points if $\{x : |b(x)| < 1\} \subset G$. In [4], Hoffman proved that for $x \in M(H^\infty)$, $x \in G$ if and only if $x \in Z(b)$ for some interpolating Blaschke product b . He also proved that for a point $x \in G$, there exists a one-to-one continuous onto map $L_x : D \rightarrow P(x)$ such that $L_x(0) = x$ and $f \circ L_x \in H^\infty$ for every $f \in H^\infty$. The map L_x , which is called the Hoffman map for the point x , is given by

$$L_x(z) = \lim_{\alpha} \frac{z + z_0}{1 + \bar{z}_\alpha z}, \quad z \in D, \quad (1.7)$$

where $\{Z_\alpha\}_\alpha$ is a net in D which converges to x . A part $P(x)$ is called sparse if there is a sparse Blaschke product b such that $b(x) = 0$. In this case we have $|(b \circ L_x)'(0)| = 1$. Therefore, b is a sparse Blaschke product if and only if $|(b \circ L_x)'(0)| = 1$ for every $x \in Z(b)$. A part is called locally sparse if there is an interpolating Blaschke product b such that $b(x) = 0$ and $|(b \circ L_x)'(0)| = 1$.

For an interpolating Blaschke product b with zeros $\{z_n\}_n$, let

$$\delta_0(b) = \lim_{n \rightarrow \infty} \inf_{k \neq n} \rho(z_n, z_k). \quad (1.8)$$

An interpolating Blaschke product b is called spreading if $\delta_0(b) = 1$. By considering boundary function, we may consider H^∞ , as a closed subalgebra of L_∞ , the Banach algebra of essentially bounded Lebesgue measurable functions on the unit circle T . It is known that $M(L^\infty) \subset M(H^\infty)$ and $M(L^\infty)$ is the Shilov boundary for H^∞ . Any uniformly closed subalgebra B with $H^\infty \subset B \subset L^\infty$ is called a Douglas algebra. For a point $x \in M(H^\infty)$, there exists a probability measure μ_x on $M(L^\infty)$ such that

$$f(x) = \int_{M(L^\infty)} f d\mu_x \quad \forall f \in H^\infty. \quad (1.9)$$

We denote by $\text{supp } \mu_x$ the closed support set of μ_x . Since $\text{supp } \mu_x$ is a weak peak set of $M(L^\infty)$ for H^∞ , we have $H^\infty_{\text{supp } \mu_x} = \{f \in L^\infty : f|_{\text{supp } \mu_x} \in H^\infty_{\text{supp } \mu_x}\}$ is a Douglas algebra.

For $E \subset M(H^\infty)$, a point $x \in E$ is called a minimal support point for E if

$$\text{supp } \mu_x \subset \text{supp } \mu_y \quad \text{or} \quad \text{supp } \mu_x \cap \text{supp } \mu_y = \emptyset \quad \forall y \in E. \quad (1.10)$$

If x is a minimal support point for E , $\text{supp } \mu_x$ is called a minimal support set for E . For an interpolating Blaschke product b , we denote by $m(Z(b))$ the set of minimal support points for the set $\{x : |b(x)| < 1\}$. Let X be a Banach algebra with identity and let B be a closed subalgebra of X . The Bourgain algebra B_b of B relative to X is defined by the set of f in X such that $\|f f_n + B\| \rightarrow 0$ for every sequence $\{f_n\}_n$ in B with $f_n \rightarrow 0$ weakly. If A and B are Douglas algebras with $A \subseteq B$ and properly contained, then B is a minimal superalgebra of A if and only if $\text{supp } \mu_x = \text{supp } \mu_y$ for every $x, y \in M(A) \setminus M(B)$. We denote by B_m the smallest Douglas algebra which contains all minimal superalgebras of B . We note that $B_b \subset B_m$. An interpolating Blaschke product b such that $\{x : |b(x)| < 1\} \subset G$, with $Z(b) \cap P(x)$ being a finite set for every $x \in Z(b)$, is said to be of finite type.

2. Proofs of the theorems

THEOREM 2.1. *An interpolating Blaschke product b that is unimodular on trivial parts is of finite type if and only if $m(Z(b)) = \{z : |b(z)| < 1\}$.*

PROOF. Suppose b is an interpolating Blaschke product that is unimodular on the trivial points and of finite type. Let $z \in M(H^\infty + C)$ such that $|b(z)| < 1$. By [1, Theorems 1 and 2], there is an $x \in m(Z(b))$ such that $\text{supp } \mu_x \subset \text{supp } \mu_z$. By [3, Theorem 3.1], the set $\text{supp } \mu_x$ is a maximal support set. Hence $\text{supp } \mu_x = \text{supp } \mu_z$. This implies that z is a minimal support point for b , that is, $z \in m(Z(b))$. So $\{z : |b(z)| < 1\} \subset m(Z(b))$. Since $m(Z(b)) \subset \{z : |b(z)| < 1\}$, we have $\{z : |b(z)| < 1\} = m(Z(b))$. Conversely, suppose $m(Z(b)) = \{z : |b(z)| < 1\}$ and assume that b is unimodular on trivial points but not of finite type. Then there is a $y \in Z(b)$ such that the set $Z(b) \cap P(y)$ is an infinite set. By [2, Theorems 1 and 2], there is an $x \in M(H^\infty + C)$ such that $|b(x)| < 1$, an uncountable index set I such that for $\alpha, \beta \in I$, $\alpha \neq \beta$, $\text{supp } \mu_{x_\alpha} \cap \text{supp } \mu_{x_\beta} = \emptyset$, $x_\alpha, x_\beta \in m(Z(b))$, and $\text{supp } \mu_{x_\alpha} \subset \text{supp } \mu_x$ for all $\alpha \in I$. Since $\text{supp } \mu_{x_\alpha}$ is properly contained in $\text{supp } \mu_x$, this implies that $x \notin m(Z(b))$ but $|b(x)| < 1$. This contradicts our assumption that $\{z : |b(z)| < 1\} = m(Z(b))$. Thus, b is of finite type. $\square$

THEOREM 2.2. *Suppose that b is a spreading nonsparse Blaschke product, and $x \in m(Z(b))$ such that $|(b \circ L_x)'(0)| \neq 1$. Then*

- (i) $(H_{\text{supp } \mu_x}^\infty)_b = H_{\text{supp } \mu_x}$,
- (ii) $(H_{\text{supp } \mu_x}^\infty)_m = H_{\text{supp } \mu_x}^\infty[\tilde{b}]$.

PROOF. By [5, Lemma 2.1], we have that P_x is a nonlocally sparse part. Hence, by [6, Theorem 5] we have that (i) holds.

Since b is spreading and $x \in m(Z(b))$,

$$M(H_{\text{supp } \mu_x}^\infty) = M(H_{\text{supp } \mu_x}^\infty[\tilde{b}]) \cup E_x, \quad (2.1)$$

where $E_x = \{y \in M(H^\infty + C) : \text{supp } \mu_y = \text{supp } \mu_x\}$. This implies that $H_{\text{supp } \mu_x}^\infty$ is properly contained in $(H_{\text{supp } \mu_x}^\infty)_m$. Since $H_{\text{supp } \mu_x}^\infty$ is a maximal subalgebra of $H_{\text{supp } \mu_x}^\infty[\tilde{b}]$, $H_{\text{supp } \mu_x}^\infty[\tilde{b}]$ is contained in $(H_{\text{supp } \mu_x}^\infty)_m$. Since

$$M(H_{\text{supp } \mu_x}^\infty) = M(L^\infty) \cup \{y \in M(H^\infty + C) : \text{supp } \mu_y \subseteq \text{supp } \mu_x\}, \quad (2.2)$$

we show that if q is an interpolating Blaschke product such that $\tilde{q} \in (H_{\text{supp } \mu_x}^\infty)_m$, then $H_{\text{supp } \mu_x}^\infty[\tilde{q}] = H_{\text{supp } \mu_x}^\infty[\tilde{b}]$. This proves (ii). Suppose that we have $H_{\text{supp } \mu_x}^\infty[\tilde{b}]$ properly contained in $(H_{\text{supp } \mu_x}^\infty)_m$, then we have $M((H_{\text{supp } \mu_x}^\infty)_m)$ properly contained in $M(H_{\text{supp } \mu_x}^\infty[\tilde{b}])$. So there is a $y \in M(H_{\text{supp } \mu_x}^\infty[\tilde{b}])$, an interpolating Blaschke product q with $\tilde{q} \in (H_{\text{supp } \mu_x}^\infty)_m$ and $q(y) = 0$. By (2.2) we have $y \in M(H_{\text{supp } \mu_x}^\infty)$ but $y \notin E_x$. Again, by (2.2), this implies that $\text{supp } \mu_y$ is properly contained in the $\text{supp } \mu_x$. By [2, Theorems 1 and 2], there is an uncountable index set I such that if $\alpha, \beta \in I$, $\alpha \neq \beta$, there are $x_\alpha, x_\beta \in Z(q)$ with $\text{supp } \mu_\alpha \cap \text{supp } \mu_{x_\beta} = \emptyset$ and $\text{supp } \mu_\alpha, \text{supp } \mu_{x_\beta}$ are both properly contained in $\text{supp } \mu_x$. This implies that

$$\cup_{\alpha \in I} E_{x_\alpha} \subset \{m \in M(H_{\text{supp } \mu_x}^\infty) : |q(m)| < 1\}. \quad (2.3)$$

But this contradicts [2, Theorem 3] since $\alpha \neq \beta$ implies that $E_{x_\alpha} \cap E_{x_\beta} = \emptyset$. Thus, no such y exists and we have $H_{\text{supp } \mu_X}^\infty[\bar{b}] = H_{\text{supp } \mu_X}^\infty[\bar{q}]$. So (ii) holds. $\square$

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