REMARKS ON A PAPER BY SILVERMAN

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ABSTRACT. We improve a result in Silverman's paper (1999) and answer a question he posed. We also consider a similar problem and obtain sufficient conditions for starlikeness.

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1. Introduction. Let A be the class of analytic functions in the unit disc $U = \{z : |z| < 1\}$ having expansion of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \quad (1.1)$$

and let $S \subset A$ be the set of univalent functions in U . A function $f \in S$ is said to be starlike of order α , $0 < \alpha < 1$, and is denoted by S_α^* if $\operatorname{Re} z(f'(z)/f(z)) > \alpha$, $z \in U$ and is said to be convex and is denoted by C if $\operatorname{Re}\{1 + z(f''(z)/f'(z))\} > 0$, $z \in U$. Silverman [2] investigated properties of the functions $f \in A$ and the class

$$G_b = \left\{ f \in A \mid \left| \left(\frac{1 + z(f''(z)/f'(z))}{z(f'(z)/f(z))} \right) - 1 \right| < b, z \in U \right\}. \quad (1.2)$$

Some of the results established by him and relevant to us are given in the following theorem.

THEOREM 1.1. *Let $f \in G_b$ then*

- (i) *If $0 < b \leq 1$, $G_b \subset S^*(2/(1 + \sqrt{1 + 8b}))$ and in particular $G_1 \subset S^*(1/2)$.*
- (ii) *$G_b \subset C$ for $0 < b \leq 1/\sqrt{2}$ and $G_1 \not\subset C$.*
- (iii) *For $b \geq 11.66$, $G_b \not\subset S^*(0)$ and for large enough b , $G_b \not\subset S$.*

His method did not extend to $b > 1$ and he expected the order of starlikeness of G_b to decrease from $1/2$ to 0 as b increases from 1 to some value b_0 after which functions in G_b need not be starlike.

In this paper we establish the following theorems.

THEOREM 1.2. *Let $f \in G_b$, $0 < b \leq 1$, then $G_b \subset S^*(1/(1 + b))$ and this order of starlikeness is sharp. Furthermore, for $b > 1$ the elements of G_b need not be regular in U .*

We notice that if we put $p(z) = z(f'(z)/f(z))$, then $p(z)$ is analytic in U with $p(0) = 1$ and G_b gets transformed to

$$G_b = \left\{ f \in A, p(z) = z \frac{f'(z)}{f(z)} \mid \left| z \frac{p'(z)}{p^2(z)} \right| < b, z \in U \right\}. \quad (1.3)$$

DEFINITION 1.3. An analytic function $f(z)$ is said to be subordinate to another analytic function $g(z)$, denoted symbolically as $f(z) < g(z)$, if $f(0) = g(0)$ and there exists an analytic function $\omega(z) \in A$, $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in U$ such that $f(z) = g(\omega(z))$.

THEOREM 1.4. Let $-1 \leq \alpha \leq 1$, $0 \leq a < 1$, $\lambda > 0$ and let $p(z)$ be an analytic function in U , $p(0) = 1$, $p(z) \neq 0$, $z \in U$ satisfy the subordination

$$z \frac{p'(z)}{p^2(z)} < \frac{\lambda z}{(1+az)^{1+\alpha}}. \quad (1.4)$$

Then

$$\begin{aligned} \frac{1}{p(z)} &< 1 - \frac{\lambda}{a\alpha} (1 - (1+az)^{-\alpha}), \quad \alpha \neq 0, \\ \operatorname{Re} \frac{1}{p(z)} &> 0 \quad \text{if } 0 < \lambda \leq \frac{a\alpha}{1 - (1+a)^{-\alpha}}, \quad \alpha \neq 0. \end{aligned} \quad (1.5)$$

For $\alpha = 0$ and $0 < \lambda \leq a/\log(1+a)$

$$p(z) < \frac{1}{1 - (\lambda/a) \log(1+az)}, \quad \operatorname{Re} p(z) > \left(1 - \frac{\lambda}{a} \log(1-a)\right)^{-1}. \quad (1.6)$$

The special case of (1.4) for $\alpha = 1$, $\lambda = b - a$, $-1 \leq b < a \leq 1$ had been considered in [1]. Silverman's case corresponds to $\alpha = -1$.

In the notation of subordination the class G_b defined by (1.3) can equivalently be written as

$$G_b = \left\{ f \in A, p(z) = z \frac{f'(z)}{f(z)} \mid z \frac{p'(z)}{p^2(z)} < bz, z \in U \right\}. \quad (1.7)$$

We need the following result from [3].

THEOREM 1.5. If h is starlike in U , $h(0) = 0$ and p is analytic in U , $p(0) = 1$ satisfies

$$zp'(z) < h(z), \quad (1.8)$$

then

$$p(z) < q(z) = 1 + \int_0^z \frac{h(t)}{t} dt, \quad (1.9)$$

where q is a convex function.

2. Proof of Theorem 1.2. From (1.7), $f \in G_b$ is equivalent to

$$z \frac{p'(z)}{p^2(z)} = b\omega(z), \quad \omega(0) = 0, \quad |\omega(z)| < 1. \quad (2.1)$$

By integration from 0 to z and using $p(0) = 1$, we get

$$\frac{1}{p(z)} = 1 - b \int_0^1 \frac{\omega(tz)}{t} dt. \quad (2.2)$$

From (2.2) using Schwartz lemma for $\omega(z)$, we get

$$\left| 1 - \frac{1}{p(z)} \right| \leq b|z|, \quad (2.3)$$

or equivalently, $|z| = r$ and

$$|p^2(z)| - 2\operatorname{Re} p(z) + 1 \leq b^2 r^2 |p^2(z)|. \quad (2.4)$$

Therefore, if $b \leq 1$,

$$(1 - b^2 r^2) |p^2(z)| - 2\operatorname{Re} p(z) + 1 \leq 0. \quad (2.5)$$

This is equivalent to

$$\left| p(z) - \frac{1}{1 - b^2 r^2} \right| \leq \frac{br}{1 - b^2 r^2}, \quad \text{if } 0 \leq b \leq 1, \quad (2.6)$$

$$\left| p(z) + \frac{1}{b^2 - 1} \right| \geq \frac{b}{b^2 - 1}, \quad \text{if } b > 1. \quad (2.7)$$

Equation (2.6) gives

$$\operatorname{Re} p(z) \geq \frac{1}{1 + b} \quad (2.8)$$

and this is sharp because

$$p(z) = \frac{1}{1 + bz} \Rightarrow f(z) = \frac{z}{1 + bz} \quad (2.9)$$

satisfies (2.6). The function $p(z)$ given by (2.9) satisfies (1.7) even for $b > 1$. However, (2.9) shows that for $b > 1$ both $p(z)$ and $f(z)$ have a pole at $z = -1/b$ and $\operatorname{Re} p(z)$ can be negative. Thus, the functions $f \in G_b$ for $b > 1$ need not even be regular. $\square$

3. Proof of Theorem 1.4. We notice that the function $z/(1 + az)^{1+\alpha}$, $0 \leq a < 1$, is starlike for $0 < \alpha \leq 1$ and convex for $-1 \leq \alpha \leq 0$. Since every convex function is starlike, we obtain, from (1.4) and Theorem 1.5,

$$\begin{aligned} \frac{1}{p(z)} &< 1 - \frac{\lambda}{a\alpha} (1 - (1 + az)^{-\alpha}), \quad \alpha \neq 0, \\ \frac{1}{p(z)} &< 1 - \frac{\lambda}{a} \log(1 + az), \quad \alpha = 0. \end{aligned} \quad (3.1)$$

As $(1 + az)^{-\alpha}$, $|\alpha| \leq 1$, $\alpha \neq 0$, is a convex function with real coefficients, we obtain

$$\begin{aligned} \operatorname{Re} \frac{1}{p(z)} &> 0 \quad \text{if } 0 < \lambda \leq \frac{a\alpha}{1 - (1 + a)^{-\alpha}}, \quad |\alpha| \leq 1, \quad \alpha \neq 0, \\ \operatorname{Re} \frac{1}{p(z)} &> 0 \quad \text{if } 0 < \lambda \leq \frac{a}{\log(1 + a)}, \quad \alpha = 0. \end{aligned} \quad (3.2)$$

Hence,

$$\operatorname{Re} p(z) \geq \frac{1}{1 - (\lambda/a\alpha)\{1 - (1-a)^{-\alpha}\}}, \quad \alpha \neq 0 \quad (3.3)$$

and $f(z)$ satisfying $p(z) = z(f'(z)/f(z))$ is starlike of order $1/(1 - (\lambda/a\alpha)\{1 - (1-a)^{-\alpha}\})$, $\alpha \neq 0$ and $1/(1 - (\lambda/a)\log(1-a))$ for $\alpha = 0$.

In the special case $\alpha = 1$ and $\lambda = a - b$ we obtain $\operatorname{Re} p(z) \geq (1-a)/(1-b)$, $-1 \leq b < a \leq 1$ which corresponds to the case in [1]. If $\alpha = -1$, we obtain $\operatorname{Re} p(z) > 1/(1+\lambda)$ which agrees with Theorem 1.2. $\square$

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