COMMON FIXED POINTS OF SET-VALUED MAPPINGS

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Dedicated to late P. V. Lakshmaiah

ABSTRACT. The main purpose of this paper is to obtain a common fixed point for a pair of set-valued mappings of Greguš type condition. Our theorem extend Diviccaro et al. (1987), Guay et al. (1982), and Negoescu (1989).

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1. Introduction. Greguš [4] proved the following result.

THEOREM 1.1. Let $C$ be a closed convex subset of a Banach space $X$. If $T$ is a mapping of $C$ into itself satisfying the inequality

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|y - Ty\|$$

(1.1)

for all $x, y \in C$, where $0 < a < 1$, $0 \leq c$, $0 \leq b$, and $a + b + c = 1$, then $T$ has a unique fixed point in $C$.

Mappings satisfying the inequality (1.1) with $a = 1$ and $b = c = 0$ is called nonexpansive and it was considered by Kirk [6], whereas the mapping with $a = 0$, $b = c = 1/2$ by Wong [13]. Recently, Fisher et al. [3], Diviccaro et al. [2], Mukherjee et al. [9], and Murthy et al. [10] generalized Theorem 1.1 in many ways. In this context, we prove a common fixed point theorem for set-valued mappings using Greguš type condition. Before presenting our main theorem we need the following definitions and lemma for our main theorem.

Let $(X, d)$ be a metric space and $CB(X)$ be the class of nonempty closed bounded subsets of $X$. For any nonempty subsets $A, B$ of $X$ we define

$$D(A, B) = \inf \{d(a, b) : a \in A, b \in B\},$$

$$H(A, B) = \max \{\sup \{D(a, b) : a \in A\}, \sup \{D(a, b) : b \in B\}\}.$$  

(1.2)

The space $CB(X)$ is a metric space with respect to the above defined distance function $H$ (see Kuratowski [7, page 214] and Berge [1, page 126]). Nadler [11] has defined the contraction mapping for set-valued mappings. A set-valued mapping $F : X \to CB(X)$ is said to be contraction if there exists a real number $k$, $0 \leq k < 1$ such that $H(Fx, Fy) \leq k \cdot d(x, y)$, for all $x, y \in X$.

Throughout this paper $C(X)$ stands for a class of nonempty compact subset of $X$, $D(A, B)$ is the distance between two sets $A$ and $B$.

The following Definitions 1.2, 1.3, 1.4, and 1.5 are given in [5].
**Definition 1.2.** An orbit for a set-valued mapping \( F: X \to CB(X) \) at a point \( x_0 \) is a sequence \( \{x_n\} \), where \( x_n \in Fx_{n-1} \) for all \( n \).

**Definition 1.3.** For two set-valued mappings \( S \) and \( T: X \to CB(X) \), we define an orbit at a point \( x_0 \in X \), if there exists a sequence \( \{x_n\} \) where \( x_n \in Sx_{n-1} \) or \( x_n \in Tx_{n-1} \) depending on whether \( n \) is even or odd.

**Definition 1.4.** The metric space \( X \) is said to be \( x_0 \)-jointly orbitally complete, if every Cauchy sequence of each orbit at \( x_0 \) is convergent in \( X \).

**Definition 1.5.** Let \( F: X \to CB(X) \) be continuous. Then the mapping \( x \to d(x,Fx) \) is continuous for all \( x \in X \).

**Definition 1.6** [11]. If \( A,B \in C(X) \) then for all \( a \in A \), there exists a point \( b \in B \) such that \( d(a,b) \leq H(A,B) \).

**Lemma 1.7** [8]. Suppose that \( \phi \) is a mapping of \([0, \infty) \) into itself, which is nondecreasing, upper-semicontinuous and \( \phi(t) < t \) for all \( t > 0 \). Then \( \lim_{n \to \infty} \phi^n(t) = 0 \), where \( \phi^n \) is the composition of \( \phi \) \( n \) times.

2. Main result

**Theorem 2.1.** Let \( S \) and \( T \) be mappings of a metric space \( X \) into \( C(X) \) and let \( X \) be \( x_0 \)-jointly orbitally complete for some \( x_0 \in X \). Suppose that \( p > 0 \) and for all \( x,y \in X \) satisfying:

\[
H^p(Sx,Ty) \leq \phi(ad^p(x,y) + (1-a)\max\{D^p(x,Sx),D^p(y,Ty)\}), \tag{2.1}
\]

where \( a \in (0,1) \) and \( \phi: [0, \infty) \to [0, \infty) \) is nondecreasing, upper-semicontinuous and \( \phi(t) < t \) for all \( t > 0 \). Then \( S \) and \( T \) have a common fixed point in \( X \).

**Proof.** Let \( x_0 \in X \). For any \( x_1 \in SX_0 \), then by **Definition 1.6**, there exists a point \( x_2 \in TX_1 \) such that \( d(x_1,x_2) \leq H(Sx_0,Tx_1) \). The choice of the sequence \( \{x_n\} \) in \( X \) guarantees that

\[
x_n \in Sx_{n-1} \quad \text{if } n \text{ is even}, \quad x_n \in Tx_{n-1} \quad \text{if } n \text{ is odd}. \tag{2.2}
\]

Now, we claim that \( d(x_1,x_2) \leq d(x_0,x_1) \). Suppose \( d(x_1,x_2) > d(x_0,x_1) \) and \( \epsilon = d(x_1,x_2) \). Then by using \((2.1)\) it follows that

\[
\epsilon = d(x_1,x_2) \leq H(Sx_0,Tx_1) \\
\leq [\phi(ad^p(x_0,x_1) + (1-a)\max\{D^p(x_0,Sx_0),D^p(x_1,Tx_1)\})]^{1/p} \\
\leq [\phi(a\epsilon^p + (1-a)\epsilon^p)]^{1/p} \\
\leq [\phi(\epsilon^p)]^{1/p} < \epsilon, \quad \text{a contradiction.} \tag{2.3}
\]

Therefore \( d(x_1,x_2) \leq d(x_0,x_1) \) and

\[
d^p(x_1,x_2) \leq H^p(Sx_0,Tx_1) \\
\leq \phi(ad^p(x_0,x_1) + (1-a)\max\{D^p(x_0,Sx_0),D^p(x_1,Tx_1)\}) \tag{2.4}
\]

\[
\leq \phi(d^p(x_0,x_1)).
\]
Similarly, we have \( d^p(x_2, x_3) \leq \phi(d^p(x_1, x_2)) \leq \phi^2(d^p(x_0, x_1)) \).

Proceeding in this way, we have

\[
d^p(x_n, x_{n+1}) \leq \phi^n(d^p(x_0, x_1)) \quad \text{for } n = 0, 1, 2, \ldots
\] (2.5)

By Lemma 1.7, it follows that \( \lim_{n \to \infty} d^p(x_n, x_{n+1}) = 0 \), that is,

\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = 0
\] (2.6)

In order to prove that \( \{x_n\} \) is a Cauchy sequence, it is sufficient to show that \( \{x_{2n}\} \) is a Cauchy sequence. Suppose that \( \{x_{2n}\} \) is not a Cauchy sequence. Then there is an \( \varepsilon > 0 \) such that for a sequence of even integers \( \{n(k)\} \) defined inductively with \( n(1) = 2 \) and \( n(k+1) \) is the smallest even integer greater than \( n(k) \) such that

\[
d(x_{n(k+1)}, x_{n(k)}) > \varepsilon.
\] (2.7)

So that

\[
d(x_{n(k+1)-2}, x_{n(k)}) \leq \varepsilon.
\] (2.8)

It follows that

\[
\varepsilon < d(x_{n(k+1)}, x_{n(k)})
\]

\[
\leq d(x_{n(k+1)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)-2}) + d(x_{n(k)-2}, x_{n(k)})
\] (2.9)

for \( k = 1, 2, 3, \ldots \) Using (2.6) and (2.8) it follows that

\[
\lim_{k \to \infty} d(x_{n(k+1)}, x_{n(k)}) = \varepsilon.
\] (2.10)

By the triangle inequality, we have

\[
|d(x_{n(k+1)}, x_{n(k)}) - d(x_{n(k)}, x_{n(k)-1})| \leq d(x_{n(k+1)}, x_{n(k)-1}),
\]

\[
|d(x_{n(k)-1}, x_{n(k)+1}) - d(x_{n(k+1)}, x_{n(k)})| \leq d(x_{n(k+1)}, x_{n(k)-1}).
\] (2.11)

It follows from (2.6) and (2.10) that

\[
\lim_{k \to \infty} d(x_{n(k)}, x_{n(k)-1}) = \lim_{k \to \infty} d(x_{n(k+1)-1}, x_{n(k)+1}) = \varepsilon.
\] (2.12)

Using (2.6), we have

\[
D(x_{n(k+1)}, x_{n(k)}) \leq d(x_{n(k+1)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})
\]

\[
\leq H(Sx_{n(k+1)-1}, Tx_{n(k)}) + d(x_{n(k)+1}, x_{n(k)})
\] (2.13)

and using (2.1), we have

\[
H^p(Sx_{n(k+1)-1}, Tx_{n(k)})
\]

\[
\leq \phi(ad^p(x_{n(k)+1}, x_{n(k)}) + (1-a) \max \{D^p(x_{n(k+1)-1}, Sx_{n(k+1)-1}), D^p(x_{n(k)}, Tx_{n(k)})\}).
\] (2.14)

Using (2.8), (2.10), (2.13), (2.14), and upper semi-continuity of \( \phi \) it follows by letting \( k \to \infty \) that

\[
\varepsilon \leq [\phi(a \varepsilon^p)]^{1/p} \leq [\phi(\varepsilon^p)]^{1/p} < \varepsilon,
\] (2.15)
a contradiction. Therefore, \( \{x_{2n}\} \) is a Cauchy sequence in \( X \) and since \( X \) is \( x_0 \)-jointly orbitally complete metric space, so the sequence \( \{x_n\} \) of each orbit at \( x_0 \) is convergent in \( X \). Therefore there exists a point \( z \in X \) such that \( x_0 \to z \).

Then again using (2.1), we have

\[
D^p(x_{2n-1}, Tz) \leq H^p(Sx_{2n-2}, Tz) \\
\leq \phi(aD^p(x_{2n-2}, z) + (1-a) \max\{D^p(x_{2n-2}, Sx_{2n-2}), D^p(z, Tz)\})
\]  
(2.16)

or equivalent to

\[
D^p(x_{2n-1}, Tz) \leq \phi(aD^p(x_{2n-2}, z) + (1-a) \max\{D^p(x_{2n-2}, Sx_{2n-2}), D^p(z, Tz)\}).
\]  
(2.17)

Now taking \( n \to \infty \) in (2.17), then we have \( D^p(z, Tz) \leq \phi((1-a)D^p(z, Tz)) \) if \( z \notin Tz \), a contradiction. Thus \( z \in Tz \).

Similarly, we show that \( z \in Sz \). Hence, \( z \in Sz \cap Tz \). This completes the proof.

**Open problem.** What further restrictions are necessary for the convergence of the sequence \( \{x_n\} \) if \( \phi \) is dropped from (2.1)?

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**References**


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