SOME REMARKS ON THE INVARIANT SUBSPACE PROBLEM
FOR HYPONORMAL OPERATORS

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ABSTRACT. We make some remarks concerning the invariant subspace problem for hyponormal operators. In particular, we bring together various hypotheses that must hold for a hyponormal operator without nontrivial invariant subspaces, and we discuss the existence of such operators.

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Let $\mathcal{H}$ be a separable, infinite-dimensional, complex Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all linear and bounded operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called hyponormal (notation: $T \in H(\mathcal{H})$) if $[T^*, T] := T^* T - TT^* \succeq 0$, or equivalently, if $\|T^* x\| \leq \|Tx\|$ for every $x \in \mathcal{H}$.

The purpose of this paper is to use several results that may be applied to the invariant subspace problem (ISP) for hyponormal operators and thus to bring into focus what remains to be done to solve the problem completely. We begin by recalling some standard notation and terminology to be used. For a (nonempty) compact subset $K \subset \mathbb{C}$, we denote by $C(K)$ the Banach algebra of all continuous complex-valued functions on $K$ with the supremum norm, by $\text{Rat}(K)$ the subalgebra of $C(K)$ consisting of all rational functions with poles off the set $K$, and by $R(K)$ the closure in $C(K)$ of $\text{Rat}(K)$. For $T \in \mathcal{L}(\mathcal{H})$, the spectrum of $T$ is denoted by $\sigma(T)$ and the algebra $\{r(T) : r \in \text{Rat}(\sigma(T))\}$ by $\text{Rat}(T)$. The rational cyclic multiplicity of $T$ (notation: $m(T)$) is the smallest cardinal number $m$ with the property that there are $m$ vectors $\{x_i\}_{0 \leq i < m}$ in $\mathcal{H}$ such that $\cup \{Ax_i \mid 0 \leq i < m, A \in \text{Rat}(T)\} = \mathcal{H}$. For a bounded (nonempty) open subset $U \subset \mathbb{C}$, one denotes by $H^\infty(U)$ the Banach algebra of those analytic complex-valued functions on $U$ with the property that $\|f\|_{\infty, U} := \sup_{z \in U} |f(z)| < \infty$. The ideal of all compact operators on $\mathcal{H}$ will be denoted by $\mathcal{K} = \mathcal{K}(\mathcal{H})$. Since $\mathcal{K}$ is a two-sided, norm-closed ideal in $\mathcal{L}(\mathcal{H})$, the quotient algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}$ is a $C^*$-algebra, which is called the Calkin algebra, and the quotient map from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H})/\mathcal{K}$ will be denoted by $\pi$. For $T$ in $\mathcal{L}(\mathcal{H})$, we write $\sigma_r(T)$ (resp., $\sigma_{le}(T), \sigma_{re}(T)$) for the essential (resp., right-, left-essential) spectrum of $T$ (i.e., the spectrum (resp., right, left spectrum) of $\pi(T)$).

An operator $A \in \mathcal{K}(\mathcal{H})$ is called a trace-class operator (notation: $A \in \mathcal{C}_1(\mathcal{H})$) if the series $\text{tr}(A) := \sum_{i \in \mathbb{N}} |A(e_i, e_i)|$ is convergent, where $|A| = (A^* A)^{1/2}$ and $\{e_i\}_{i \in \mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$. An operator $A \in \mathcal{K}(\mathcal{H})$ is Hilbert-Schmidt operator (notation: $A \in \mathcal{C}_2(\mathcal{H})$) if $A^* A \in \mathcal{C}_1(\mathcal{H})$. For a selfadjoint operator $A \in \mathcal{L}(\mathcal{H})$, $A_-$ will denote its negative part $(|A| - A)/2$. Finally, $\mu$ will denote planar Lebesgue measure defined on the Borel subsets of $\mathbb{R}^2$. 
A beautiful generalization of the Berger-Shaw inequality for hyponormal operators [3] was given by Voiculescu.

**Theorem 1** (see [17]). If \([T^*, T]_+ \in \mathcal{C}_1(\mathcal{H})\) and \(X \in \mathcal{C}_2(\mathcal{H})\) is such that \(m(T + X) < +\infty\), then \([T^*, T]_+ \in \mathcal{C}_1(\mathcal{H})\) and \(\pi \text{ tr}([T^*, T]) \leq m(T + X)\mu(\sigma(T + X))\).

For purposes of finding a nontrivial invariant subspace (n.i.s.) for an arbitrary operator \(T \in \mathcal{L}(\mathcal{H})\), one may assume that every nonzero vector in \(\mathcal{H}\) is cyclic for \(T\), and hence that \(m(T) = 1\). Thus the following corollary (of Theorem 1 or the earlier Berger-Shaw inequality) is useful.

**Corollary 2.** Every hyponormal operator \(T\) in \(\mathcal{L}(\mathcal{H})\) with a rational cyclic vector (i.e., \(m(T) = 1\)) is essentially normal (i.e., \(\pi(T)\) is normal in the Calkin algebra).

When looking for a n.i.s. for an arbitrary \(T\) in \(\mathcal{L}(\mathcal{H})\), one knows (cf. [10]) from a deep theorem of Apostol, Foiaş, and Voiculescu [2] that \(T\) may be assumed to belong to the class \(\mathcal{B}_{\mathcal{T}}(\mathcal{H})\) of biqr southwestern operators, (see [10] for the definition and a characterization). If we denote by \(\mathcal{E}_N(\mathcal{H})\) the collection of essentially normal operators on \(\mathcal{H}\) and by \((N + K)(\mathcal{H})\) the collection of those operators in \(\mathcal{L}(\mathcal{H})\) which can be written as a sum of a normal and a compact operator, then a deep result of Brown-Douglas-Fillmore can also be applied.

**Theorem 3** (see [5]). The equality \(\mathcal{B}_{\mathcal{T}}(\mathcal{H}) \cap \mathcal{E}_N(\mathcal{H}) = (N + K)(\mathcal{H})\) holds.

It is thus immediate from Theorems 1 and 3 and Corollary 2 that if there exists \(T\) in \(H(\mathcal{H})\) without a n.i.s., then \([T^*, T]_+ \in \mathcal{C}_1(\mathcal{H})\) and \(T \in (N + K)(\mathcal{H})\).

A hyponormal operator is called pure if it does not have a nonzero reducing subspace to which its restriction is a normal operator. Obviously an operator in \(H(\mathcal{H})\) without a n.i.s. is pure. The following result of Putnam [14] leads to another reduction of the ISP for hyponormal operators.

**Theorem 4.** Let \(T \in \mathcal{L}(\mathcal{H})\) be a pure hyponormal operator. If \(\Delta \subset \mathbb{C}\) is an open set, then \(\mu(\Delta \cap \sigma(T)) > 0\) whenever \(\Delta \cap \sigma(T) \neq \emptyset\).

This says that each point of the spectrum of such a \(T\) has positive planar density, and thus we may assume of a hyponormal operator \(T\) without a n.i.s. that \(\sigma(T)\) has not only positive \(\mu\)-measure but positive planar density at each point.

Let \(A \in \mathcal{L}(\mathcal{H})\) be a selfadjoint operator and denote by \(E\) the spectral measure of the operator \(A\). To every vector \(x \in \mathcal{H}\) one may associate the Borel measure \(\nu_x\) on \(\mathbb{R}\) defined by \(\nu_x(\Omega) = \langle E(\Omega)x, x\rangle\) for every Borel set \(\Omega \subset \mathbb{R}\). The vector \(x\) is called absolutely continuous with respect to \(A\) if the measure \(\nu_x\) is absolutely continuous with respect to Lebesgue measure on \(\mathbb{R}\). The selfadjoint operator \(A\) is called absolutely continuous if every vector of \(\mathcal{H}\) is absolutely continuous with respect to \(A\). The following result can be found in [13], (see also [9, page 135]).

**Proposition 5.** If \(T = X + iY \in \mathcal{L}(\mathcal{H})\) is the Cartesian decomposition of a pure hyponormal operator, then \(X\) and \(Y\) are both absolutely continuous operators.

Next, recall that a subset \(\Delta\) of a nonempty open set \(U\) in \(\mathbb{C}\) is called dominating for \(U\) if \(\|f\|_\infty, U = \sup_{\lambda \in \Delta} |f(\lambda)|\), \(f \in H^\infty(\mathcal{U})\).
The deep invariant subspace theorem for hyponormal operators obtained by Brown in [6] on the basis of the beautiful structure theorem for such operators by Putinar [12] is the following.

**Theorem 6.** Let $T \in \mathcal{B}(\mathcal{K})$ be a hyponormal operator. If there is a nonempty open set $U \subset \mathbb{C}$ such that $\sigma(T) \cap U$ is dominating for $U$, then $T$ has a n.i.s.

Since one knows (cf. [1] or [6]) that if $K$ is a (nonempty) compact set in $\mathbb{C}$ such that $R(K) = C(K)$, then $K$ is dominating on some nonempty open set, one gets immediately the following corollary.

**Corollary 7** (see [6]). Any hyponormal operator $T \in \mathcal{B}(\mathcal{K})$ with $R(\sigma(T)) \neq C(\sigma(T))$ has a n.i.s.

Thus if there exist hyponormal operators $T$ without a n.i.s., then as noted above, $T \in (N + K)(\mathcal{K})$ and $\sigma(T)$ must satisfy $R(\sigma(T)) = C(\sigma(T))$. Moreover, it is a consequence of elementary Fredholm theory (cf. [10]) that if $T \in \mathcal{B}(\mathcal{K})$ and $\sigma_{le}(T) \neq \sigma(T)$, then $T^*$ has point spectrum and thus $T$ has a n.i.s. Hence when looking for invariant subspaces for an arbitrary operator $T$ we may always suppose that $\sigma_{le}(T) = \sigma_{re}(T) = \sigma(T)$. This allows one to apply a result of Stampfli [16] to the problem.

**Theorem 8.** Suppose $T \in \mathbb{K}N(\mathcal{K})$ is such that $\sigma(T) = \sigma_{le}(T)$ and the Calkin map $\pi : \text{Rat}(T) \to \mathcal{B}(\mathcal{K})/\mathbb{K}$ is bounded below. Then $T$ has a n.i.s.

**Proof.** By hypothesis, there exists a constant $M > 0$ such that $\|\pi(r(T))\| \geq M\|r(T)\|$ for every $r \in \text{Rat}(\sigma(T))$, where $\| \cdot \|_e$ is the norm in the Calkin algebra. On the other hand,

$$\|\pi(r(T))\|_e = \|\pi(T)\|_e = \sup_{z \in \sigma(T)} r(z) = \sup_{z \in \sigma(T)} |r(z)|.$$

Thus $\|r(T)\| \leq (1/M)\|r|_{\sigma(T)}$, $r \in \text{Rat}(\sigma(T))$, so $\sigma(T)$ is a $(1/M)$-spectral set for $T$. The result now follows from [16].

**Corollary 9.** If $T \in H(\mathcal{K})$ and $T$ has no n.i.s., then there exist sequences $\{r_n(T)\}_{n \in \mathbb{N}}$ in the algebra $\text{Rat}(T)$ and $\{K_n\}_{n \in \mathbb{N}}$ in $\mathbb{K}$ such that $\|r_n(T)\| = 1$, $n \in \mathbb{N}$, and $\|r_n(T) - K_n\| \to 0$. Moreover, any such sequence $\{r_n(T)\}_{n \in \mathbb{N}}$ has no subnet converging in the weak operator topology (WOT) to a nonzero operator.

**Proof.** Since $T \in H(\mathcal{K})$ has no n.i.s., $m(T) = 1$, and according to Corollary 2, $T \in \mathbb{K}N(\mathcal{K})$. According to Theorem 8, $\pi : \text{Rat}(T) \to \mathcal{B}(\mathcal{K})/\mathbb{K}$ must not be bounded below. Thus, there are sequences $\{r_n(T)\}_{n \in \mathbb{N}}$ in the algebra $\text{Rat}(T)$ and $\{K_n\}_{n \in \mathbb{N}}$ in $\mathbb{K}$ such that $\|r_n(T)\| = 1$, $n \in \mathbb{N}$, and $\|r_n(T) - K_n\| \to 0$. Moreover, if there is a subnet $\{r_{nk}(T)\}_{k \in \mathbb{N}}$ converging in the WOT to a nonzero operator, then $T$ has a nontrivial hyperinvariant subspace according to [8].

The following proposition simply summarizes the results mentioned above.

**Proposition 10.** If there exists $T \in H(\mathcal{K})$ such that $T$ has no n.i.s., then $T$ has the following properties:

(a) $\sigma_{le}(T) = \sigma_{re}(T) = \sigma(T)$,
whose spectrum is each point, there is an irreducible, hyponormal operator with rank one self-commutator
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0 < t_0 < 1. Clearly there exist positive real numbers t_1, t_2 such that 0 < t_1 < t_0 < t_2 < 1 and some nondegenerate interval [a, b] with p_0 ∈ [a, b] such that the trapezoid-like figure Γ := \{tp + (1-t)z_0 : t_1 ≤ t ≤ t_2, p ∈ C(θ_1) \cap [a, b]\} is contained in ∆. Let α > 0 be the linear measure of the set C(θ_1) \cap [a, b]. Then the intersection of the set K_1 ∩ Γ with each horizontal line y = t, t_1 ≤ t ≤ t_2 has (linear) measure (1-t)α. Thus, by Fubini’s theorem, the μ-measure of the set K_1 ∩ Γ is \int_{t_1}^{t_2} (1-t)α dt = (t_2-t_1)(1-(t_1+t_2)/2)α > 0, and hence K_1 has property (b). To check that K_1 has property (c), the following lemma is useful.

**Lemma 14.** If U is any bounded open set such that U ∩ K_1 ≠ ∅, then there exist a point p_0 belonging to the outer boundary of U, an ε > 0, and a disc D = \{z ∈ C : |z - p_0| < ε\} such that D ∩ K_1 = ∅.

**Proof.** By construction, [0, 1] \ C(θ_n) = ∪_{n=1}^{∞} (a_n, b_n) where \{(a_n, b_n)\}_{n=1}^{∞} is the disjoint sequence of “excluded” open intervals. Thus each open triangular domain T_n = \{z_0 + (1-t)p : -∞ < t < 1, p ∈ (a_n, b_n)\} in C is disjoint from K_1. Since U ∩ K_1 ≠ ∅ and every point of C(θ_1) is a limit point of end points of arbitrarily short excluded intervals, there exists some triangular domain T_n₀ such that U ∩ T_n₀ ≠ ∅, and clearly any half-line joining a point of U ∩ T_n₀ to the ideal point |z| = +∞ and lying entirely in T_n₀ must intersect ∂U in some last point (since ∂U is compact), which clearly satisfies the desired conclusions. 

We next show that K_1 satisfies (c) of Proposition 10. Let U be a bounded open set in C such that U ∩ K_1 ≠ ∅ and set C = U⁻. Then the outer boundary of C coincides with the outer boundary of U, and applying Lemma 14 to U, we get a point p₀ of the outer boundary of U and an open disc D centered at p₀ with radius ε > 0 such that D ∩ K_1 = ∅. By [7, Corollary 13.3], p₀ is a peak point of R(C), that is, there exists an f₀ ∈ R(C) such that f₀(p₀) = 1 and |f₀(z)| < 1 for z ∈ C \ \{p₀\}. Clearly f₀ ∈ H∞(C) and sup_{λ ∈ U} |f₀(λ)| = 1 (since p₀ ∈ ∂U) while sup_{λ ∈ K_1 ∩ U} |f₀(λ)| < 1 (since (K_1 ∩ U)⁻ is at positive distance from p₀ and |f₀| < 1 on (K_1 ∩ U)⁻ ⊂ C \ D). Thus K_1 ∩ U is not dominating for U, and K_1 has property (c).

Let T₁ be an irreducible hyponormal operator with rank one self-commutator whose spectrum σ(T₁) is the compact set K₁ described in Example 13, and whose existence is guaranteed by Proposition 12. Thus property (f) is also satisfied. One observes that property (a) is satisfied too. Indeed, if one assumes that there exists λ₀ ∈ σ(T₁) \ σ_{re}(T₁), then T₁ - λ₀ is semi-Fredholm operator with nonpositive index (since T₁ ∈ H(π)). Since T₁ is pure, σ_p(T₁) = ∅, and thus the index is negative, which implies that σ(T₁) contains a nonempty open set. Obviously this is a contradiction since σ(T₁) = K₁ has no interior, and thus σ(T₁) = σ_{re}(T₁). In a similar way one shows that σ(T₁) = σ_{re}(T₁). Moreover, T₁ ∈ H(π) according to theorem of Apostol, Foiaș, and Voiculescu [2] since σ_{e}(T₁) contains no pseudoholes or holes associated with Fredholm index different from 0. Thus, by Theorem 3, the operator T₁ can be written T₁ = N + K, where N is a normal operator and K is a compact operator. Moreover, since T₁ is pure, T₁ = X₁ + iY₁, where X₁ and Y₁ are absolutely continuous selfadjoint operators according to Proposition 5.

Thus we have shown that the operator T₁ has properties (a)-(f) of Proposition 10. Whether T₁ has properties (g) and (h) of this proposition the author is unable to
conclude. However, techniques and results of Stampfli [15] may be applied to obtain the following theorem.

**THEOREM 15.** Any hyponormal operator $T$ in $\mathcal{H}$ such that $\sigma(T)$ is the set $K_1$ of Example 13 has a nontrivial hyperinvariant subspace.

**Proof.** Consider the operator $T - (1/2 + i)$, which has spectrum $K_1 - (1/2 + i)$. Define $K_+ := K_1 \cap D_1^1$, where $D_1 := \{z \in \mathbb{C} : |z - 4| < 4\}$ and $K_- := K_1 \cap D_2^2$, where $D_2 = \{z \in \mathbb{C} : |z + 4| < 4\}$. Then $\sigma(T) = K_+ \cup K_-, K_+ \cap K_- = \{(0,0)\}$, and $\partial D_1 \cap K_1 = \partial D_2 \cap K_1 = \{(0,0)\}$. Choosing $f_1(z) = f_2(z) = z^2$, we may follow [15, Example 1] and observe that the operators $A_i := \int_{\partial D_i} f_i(z)(z - T)^{-1}dz$, $i = 1,2$, commute with any operator $S$ with which $T$ commutes to conclude that $T$ has a nontrivial hyperinvariant subspace.

**Remark 16.** We note that it is quite easy to modify the construction of Example 13 to produce compact sets satisfying properties (b) and (c) of Proposition 10 such that the techniques of [15] of “integrating through the spectrum” are no longer available to produce nontrivial invariant subspaces for the corresponding operator $T$.

**References**


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