WHITEHEAD GROUPS OF EXCHANGE RINGS WITH PRIMITIVE FACTORS ARTINIAN

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ABSTRACT. We show that if R is an exchange ring with primitive factors artinian then $K_1(R) \cong U(R)/V(R)$, where $U(R)$ is the group of units of R and $V(R)$ is the subgroup generated by $\{(1+ab)(1+ba)^{-1} \mid a, b \in R \text{ with } 1+ab \in U(R)\}$. As a corollary, $K_1(R)$ is the abelianized group of units of R if $1/2 \in R$.

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Very recently, Ara et al. [2] showed that the natural homomorphism $GL_1(R) \rightarrow K_1(R)$ is surjective provided that R is a separative exchange ring. A natural problem is the description of the kernel of the epimorphism $GL_1(R) \rightarrow K_1(R)$. In [9, Theorems 1.2 and 1.6], Menal and Moncasi showed that if R satisfies unit 1-stable range or is unit-regular, then $K_1(R) \cong U(R)/V(R)$. Here $U(R)$ is the group of units of R while $V(R)$ is a subgroup described later. In [7], Goodearl and Menal remarked that if for each $x, y \in R$ there exists a unit $u \in R$ such that $x - u$ and $y - u^{-1}$ are both units, then $K_1(R) \cong U(R)^{ab}$. In this paper, we investigate the above kernel for exchange rings with primitive factors artinian.

Recall that R is called an exchange ring if for every right R -module A and two decompositions $A = M \oplus N = \oplus_{i \in I} A_i$, where $M_R \cong R$ and the index set I is finite, there exist submodules $A'_i \subseteq A_i$ such that $A = M \oplus (\oplus_{i \in I} A'_i)$. It is well known that regular rings, π -regular rings, semiperfect rings, left or right continuous rings, clean rings and unit C^* -algebras of real rank zero [1] are all exchange rings.

Many authors have studied exchange rings with primitive factors artinian. Fisher and Snider proved that every regular ring with primitive factors artinian is unit-regular (see [6, Theorem 6.10]). Moreover, Menal [8, Theorem B] proved that every π -regular ring with primitive factors artinian has stable range one. Recently, Yu [13, Theorem 1] extended these results to exchange rings and showed that every exchange ring with primitive factors artinian has stable range one. On the other hand, Pardo [10] investigated the Grothendieck group K_0 of exchange rings. In this paper, we show that the Whitehead group $K_1(R) \cong U(R)/V(R)$ for an exchange ring R with primitive factors artinian. We refer the reader to [11] for the general theory of Whitehead groups.

Throughout this paper, all rings are associative with identity. The set $U(R)$ denotes the set of all units of R , $V(R)$ denotes the subgroup generated by $\{p(a, b)p(b, a)^{-1} \mid p(a, b) \in U(R), a, b \in R\}$, and $J(R)$ denotes the Jacobson radical of R (see below for the definition of $p(a, b)$ and other continuant polynomials). If G is a group and G' its

commutator subgroup, then G^{ab} denotes G/G' . Let $\text{GL}_n(R)$ be the group of units of $M_n(R)$, the ring of all $n \times n$ matrices over R .

We start with the following new element-wise property of exchange rings with primitive factors artinian.

LEMMA 1. *Let R be an exchange ring with primitive factors artinian. Then, for any $x, y \in R$, there exists a unit-regular $w \in R$ such that $1 + xy - xw \in U(R)$.*

PROOF. Assume that there are some $x, y \in R$ such that $1 + xy - xw \notin U(R)$ for any unit-regular $w \in R$. Let Ω be the set of all two-sided ideals A of R such that $1 + xy - xw$ is not a unit modulo A for any unit-regular $w + A \in R/A$. Clearly, $\Omega \neq \emptyset$.

Given any ascending chain $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ in Ω , set $M = \cup_{1 \leq i \leq \infty} A_i$. Then M is a two-sided ideal of R . Assume that M is not in Ω . Then there exists a unit-regular $w + M \in R/M$ such that $(1 + xy - xw) + M$ is a unit of R/M . Since $w + M$ is unit-regular in R/M , we have $e + M = (e + M)^2$, $u + M \in U(R/M)$ such that $w + M = (e + M)(u + M)$. As R/M is also exchange, we may assume that $e = e^2 \in R$. Thus we can find positive integers n_i ($1 \leq i \leq 5$) such that $(1 + xy - xw)s - 1 \in A_{n_1}$, $s(1 + xy - xw) - 1 \in A_{n_2}$, $w - eu \in A_{n_3}$, $ut - 1 \in A_{n_4}$, and $tu - 1 \in A_{n_5}$ for some $s, t \in R$. Let $n = \max\{n_1, n_2, n_3, n_4, n_5\}$. Then $1 + xy - xw + A_n$ is a unit of R/A_n for a unit-regular $w + A_n \in R/A_n$. This contradicts the choice of A_n . So $M \in \Omega$. That is, Ω is inductive. By using Zorn's lemma, we have a two-sided ideal Q of R such that it is maximal in Ω .

Let $S = R/Q$. If $J(R/Q) \neq 0$, then $J(R/Q) = K/Q$ for some $K \supset Q$. Clearly, $S/J(S) \cong R/K$. By the maximality of Q , we can find a unit-regular $(v + Q) + J(S)$ such that $((1 + xy - xv) + Q) + J(S)$ is a unit of $S/J(S)$. Since idempotents and units of $S/J(S)$ can be lifted modulo $J(S)$, we may assume that $v + Q$ is unit-regular in S . On the other hand, $(1 + xy - xv) + Q = (m + Q) + (r + Q)$ for some $m + Q \in U(S)$, $r + Q \in J(S)$. Thus, $(1 + xy - xv) + Q$ is a unit of S . This gives a contradiction, whence $J(R/Q) = 0$.

By the maximality of Q , one easily checks that R/Q is an indecomposable ring. According to [14, Lemma 3.7], R/Q is a simple artinian ring. Clearly, $(1 + xy - xy) + Q = 1 + Q$ is a unit of R/Q . Since R/Q is unit-regular, $y + Q$ is a unit-regular element of R/Q . This contradicts the choice of Q . Therefore the proof is complete. $\square$

We note that the following conditions for a ring R are equivalent.

- (1) For any $x, y \in R$, there exists unit-regular $w \in R$ such that $1 + x(y - w) \in U(R)$.
- (2) Given $aR + bR = R$, then $a + bw \in U(R)$ for some unit-regular $w \in R$.
- (3) Given $ax + b = 1$, then $aw + b \in U(R)$ for some unit-regular $w \in R$ (cf. [5, Theorem 2.1]).

Clearly, the conditions above are stronger than the stable range one condition and can be viewed as a generalization of rings with many idempotents [4]. Call R a ring with many unit-regular elements if the equivalent conditions above hold. In [5, Theorem 3.1], the authors showed that such rings are all GE_2 . Furthermore, Chen [3] proved that if R has many unit-regular elements then so does $M_n(R)$. Now we include the fact that every exchange ring with primitive factors artinian has many unit-regular elements to make our paper self-contained.

Recall that $p(a) = a$, $p(a, b) = 1 + ab$, and $p(a, b, c) = a + c + abc$ for any $a, b, c \in R$. It is easy to verify that $p(a, b, c) = p(a, b)c + p(a)$, $p(a, b, c)p(b, a) = p(a, b)p(c, b, a)$.

LEMMA 2. *Let $a, b, c \in R$ with $p(a, b, c) \in U(R)$. If a is unit-regular, then $p(a, b, c) \equiv p(c, b, a) \pmod{V(R)}$.*

PROOF. Since a is unit-regular, there is an idempotent e and a unit u such that $a = eu$. So we have $p(a, b, c) = eu + c + eubc$, and then $p(a, b, c)u^{-1} = e + cu^{-1} + eubcu^{-1}$. Obviously, $p(e, -ub(1-e)) = p(e, -ub(1-e))(p(-ub(1-e), e))^{-1} \in V(R)$. Thus we see that

$$\begin{aligned} p(a, b, c)u^{-1} &\equiv p(e, -ub(1-e))p(a, b, c)u^{-1} \pmod{V(R)} \\ &= (1 - eub(1-e))(e + cu^{-1} + eubcu^{-1}) \\ &= e + cu^{-1} + eubecu^{-1} \pmod{V(R)}. \end{aligned} \quad (1)$$

On the other hand, we can verify that

$$\begin{aligned} p(c, b, a)u^{-1} &= e + cu^{-1} + cbe \equiv (e + cu^{-1} + cu^{-1}ube)p(1-e, -ube) \pmod{V(R)} \\ &= e + cu^{-1} + cu^{-1}eube = 1 + (cu^{-1} - (1-e))(1 + eube) \\ &\equiv 1 + (1 + eube)(cu^{-1} - (1-e)) \pmod{V(R)} \\ &\equiv p(a, b, c)u^{-1} \pmod{V(R)}. \end{aligned} \quad (2)$$

Consequently, $p(a, b, c) \equiv p(c, b, a) \pmod{V(R)}$, as asserted. $\square$

In [4, Theorem 16], Chen showed that $K_1(R) \cong U(R)/V(R)$ provided that R has idempotent 1-stable range. Now we extend this fact to exchange rings with primitive factors artinian.

THEOREM 3. *Let R be an exchange ring with primitive factors artinian. Then $K_1(R) \cong U(R)/V(R)$.*

PROOF. Given $a, b, c \in R$ with $p(a, b, c) \in U(R)$, we have $p(c, b, a) \in U(R)$. From Lemma 1, we can find some unit-regular $w \in R$ such that $1 + b(c - w) \in U(R)$. Let $c - w = t$. Then $c = t + w$ and $1 + bt \in U(R)$. We check that

$$\begin{aligned} p(a, b, c) &= a + c + abc = (a + t + abt) + (1 + ab)w \\ &= (a + t + abt) + (1 + ab + tb + abtb)(1 + tb)^{-1}w \\ &= (a + t + abt) + (1 + tb)^{-1}w + (a + t + abt)b(1 + tb)^{-1}w \\ &= (1 + tb)^{-1}((1 + tb)(a + t + abt) + w + (1 + tb)(a + t + abt)b(1 + tb)^{-1}w) \\ &= (1 + tb)^{-1}p((1 + tb)(a + t + abt), b(1 + tb)^{-1}, w). \end{aligned} \quad (3)$$

In view of Lemma 2, we know that

$$\begin{aligned} p(a, b, c) &\equiv (1 + tb)^{-1}p(w, b(1 + tb)^{-1}, (1 + tb)(a + t + abt)) \pmod{V(R)} \\ &= (1 + tb)^{-1}(p(w, b(1 + tb)^{-1})(1 + tb)(a + t + abt) + p(w)) \\ &= (1 + tb)^{-1}(p(w, b(1 + tb)^{-1})p(t, b)p(a, b, t) + p(w)) \\ &= (1 + tb)^{-1}(p(w, b(1 + tb)^{-1})p(t, b, a)p(b, t) + p(w)). \end{aligned} \quad (4)$$

On the other hand,

$$\begin{aligned}
 p(c, b, a) &= p(w, (1+bt)^{-1}b, (t+a+tba)(1+bt))(1+bt)^{-1} \\
 &= (p(w, (1+bt)^{-1}b)(t+a+tba)(1+bt) + p(w))(1+bt)^{-1} \\
 &= (p(w, (1+bt)^{-1}b)p(t, b, a)p(b, t) + p(w))(1+bt)^{-1}.
 \end{aligned} \tag{5}$$

Since $b(1+tb)^{-1} = (1+bt)^{-1}b$, $(1+tb)p(a, b, c) \equiv p(c, b, a)(1+bt) \pmod{V(R)}$. Hence $p(a, b, c)(p(c, b, a))^{-1} \in V(R)$ because $U(R)/V(R)$ is abelian. By virtue of [13, Theorem 1], R has stable range one. It follows from [9, Theorem 1.2] that $K_1(R) \cong U(R)/W(R) \cong U(R)/V(R)$, where the notation $W(R)$ denotes the subgroup of $U(R)$ generated by $\{p(a, b, c)p(c, b, a)^{-1} \mid p(a, b, c) \in U(R), a, b, c \in R\}$. So the proof is complete. $\square$

Recall that an exchange ring is said to be of bounded index if there exists some positive integer n such that $x^n = 0$ for all nilpotent $x \in R$. We can derive the following corollary.

COROLLARY 4. *Let R be an exchange ring of bounded index. Then $K_1(R) \cong \text{GL}_3(R)^{ab} \cong U(R)/V(R)$.*

PROOF. It is easy to obtain the first isomorphism by an argument of Menal. In view of [13, Theorem 3], R is an exchange ring with primitive factors artinian. Therefore we complete the proof by Theorem 3. $\square$

Write $L_1(R)$ for the subgroup generated by the elements in $U(R)' \cup L$, where L is the subgroup generated by all $1+er(1-e)$ with $e=e^2$, $r \in R$. Clearly, $U(R)' \subseteq L_1(R) \subseteq V(R)$.

LEMMA 5. *Let $a, b \in R$ with $1+ab \in U(R)$. If a is unit-regular, then $p(a, b) \equiv p(b, a) \pmod{L_1(R)}$.*

PROOF. Since a is unit-regular, there is an idempotent e and a unit u such that $a = ue$. So we have

$$\begin{aligned}
 p(a, b) &= 1+ueb = u(1+ebu)u^{-1} \\
 &= u(1+ebu)u^{-1}(1+ebu)^{-1}(1+ebu) \\
 &\equiv (1+ebu) \pmod{L_1(R)} \\
 &\equiv (1-ebu(1-e))(1+ebu) \pmod{L_1(R)} \\
 &= 1+ebue = (1+bue)(1-(1-e)bue) \\
 &\equiv (1+bue) \pmod{L_1(R)} = p(b, a), \text{ as required.}
 \end{aligned} \tag{6}$$

LEMMA 6. *Let R be an exchange ring with primitive factors artinian. If R does not have $\mathbb{Z}/2\mathbb{Z}$ as a homomorphic image, then, for any $x, y \in R$, there exists a $w \in U(R)$ such that $1+x(y-w) \in U(R)$ and $y-w$ is unit-regular.*

PROOF. Assume that there exist some $x, y \in R$ such that $1+x(y-w) \notin U(R)$ or $y-w$ is not unit-regular for any unit $w \in R$. Let Ω be the set of all two-sided ideals A of R such that $1+x(y-w)$ is not a unit or $y-w$ is not unit-regular modulo A for any unit $w+A \in R/A$. Obviously, $\Omega \neq \emptyset$.

Analogously to the discussion in [Lemma 1](#), we can find a two-sided ideal Q of R such that it is maximal in Ω . Set $S = R/Q$. If $J(S) \neq 0$, then $J(S) = K/Q$ for some $K \supset Q$. Obviously, $S/J(S) \cong R/K$. By the maximality of Q , we have a unit $(v + Q) + J(S)$ such that $((1 + x(y - v)) + Q) + J(S)$ is a unit of $S/J(S)$ and $((y - v) + Q) + J(S)$ is unit-regular in $S/J(S)$. Since units of $S/J(S)$ can be lifted modulo $J(S)$ because S is an exchange ring, we may assume that $v + Q$ is unit of S . Similarly to [Lemma 1](#), we see that $(1 + x(y - v)) + Q = (m + Q) + (r + Q)$ for some $m + Q \in U(S)$, $r + Q \in J(S)$. Thus, $(1 + x(y - v)) + Q$ is a unit of S . On the other hand, idempotents of $S/J(S)$ can be lifted modulo $J(S)$ because S is an exchange ring. So we may assume that $((y - v) + Q) + J(S) = ((f + Q) + J(S))((u + Q) + J(S))$ with $f + Q = (f + Q)^2 \in R/Q$, $u + Q \in U(S)$. Thus we can find some $t \in R$ with $t + Q \in J(S)$ such that $(y - (v - t)) + Q = (f + Q)(u + Q)$ is unit-regular in S . From $(1 + x(y - (v - t))) + Q = ((1 + x(y - v)) + Q) + (x + Q)(t + Q)$ with $(1 + x(y - v)) + Q \in U(S)$ and $t + Q \in J(S)$, one easily checks that $(1 + x(y - (v - t))) + Q$ is a unit of S . This contradicts the choice of Q . Therefore $J(S) = 0$. The maximality of Q implies that S is indecomposable as a ring. By virtue of [\[14, Lemma 3.7\]](#), S is a simple artinian ring. Assume that $R/Q \cong M_n(D)$, where n is a positive integer and D is a division ring. Since R does not have $\mathbb{Z}/2\mathbb{Z}$ as a homomorphic image, we claim that $n \geq 2$, or $n = 1$, and $D \not\cong \mathbb{Z}/2\mathbb{Z}$. Thus R/Q satisfies unit 1-stable range. From $((1 + xy) + Q)S + ((-x) + Q)S = S$, we have a unit $s + Q$ of R/Q such that $(1 + x(y - s)) + Q = ((1 + xy) + Q) + ((-x) + Q)(s + Q)$ is a unit of S . Since S is simple artinian, $(y - s) + Q$ is unit-regular in S , a contradiction. Therefore the proof is complete. $\square$

THEOREM 7. *Let R be an exchange ring with primitive factors artinian. If R does not have $\mathbb{Z}/2\mathbb{Z}$ as a homomorphic image, then $K_1(R) \cong U(R)/L_1(R)$.*

PROOF. Since R is an exchange ring with primitive factors artinian, by virtue of [Theorem 3](#), we know that $K_1(R) \cong U(R)/V(R)$. Let $b, c \in R$ with $p(b, c) \in U(R)$. In view of [Lemma 6](#), there exists some $w \in U(R)$ such that $1 + b(c - w) \in U(R)$ and $c - w = s$ unit-regular. Observe that

$$\begin{aligned} p(b, c) &= 1 + bc = 1 + bs + bw = 1 + bs + b(1 + sb)(1 + sb)^{-1}w \\ &= (1 + bs)(1 + b(1 + sb)^{-1}w) = p(b, s)p(b(1 + sb)^{-1}, w). \end{aligned} \quad (7)$$

Likewise, we see that

$$p(c, b) = p(w, b(1 + sb)^{-1})(1 + sb) = p(w, b(1 + sb)^{-1})p(s, b). \quad (8)$$

By [Lemma 5](#) and the fact that units commute mod $L_1(R)$, we see that

$$\begin{aligned} p(b, c) &= p(b, s)p(b(1 + sb)^{-1}, w) \equiv p(b(1 + sb)^{-1}, w)p(b, s) \\ &\equiv p(w, b(1 + sb)^{-1})p(s, b) = p(c, b) \pmod{L_1(R)}. \end{aligned} \quad (9)$$

Hence, $L_1(R) = V(R)$ and we conclude that $K_1(R) \cong U(R)/L_1(R)$, as asserted. $\square$

COROLLARY 8. *Let R be an exchange ring with primitive factors artinian. If $2 \in U(R)$, then $K_1(R) \cong U(R)^{ab}$.*

PROOF. Let $e = e^2 \in R$. Since $2 \in U(R)$, we have $e = 1/2 + (2e - 1)/2$. Obviously, $((2e - 1)/2)(4e - 2) = 1$. By [\[9, Lemma 1.5\]](#), $1 + eR(1 - e) \subseteq U(R)'$, and then $L_1(R) = U(R)'$.

On the other hand, $2 \in U(R)$ implies that R does not have $\mathbb{Z}/2\mathbb{Z}$ as a homomorphic image. Using [Theorem 7](#), we conclude that $K_1(R) \cong U(R)/L_1(R) \cong U(R)/U(R)' \cong U(R)^{ab}$, as desired. $\square$

COROLLARY 9. *Let R be an exchange ring of bounded index of nilpotence. If $2 \in U(R)$, then $K_1(R) \cong U(R)^{ab}$.*

PROOF. By [\[12, Proposition 2.1\]](#), we see that the primitive factors of R are artinian. Thus the result follows from [Corollary 8](#). $\square$

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