CHARACTERIZATION OF AN H^* -ALGEBRAS IN TERMS OF A TRACE

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ABSTRACT. Arbitrary proper H^* -algebra is characterized in terms of the trace defined on a certain subset of the algebra.

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This paper deals with characterizations of an arbitrary proper (not necessarily commutative) H^* -algebra. More specifically, we show that any Banach $*$ -algebra with a partially defined trace, and some additional properties, has a Hilbert space structure with respect to which it is an H^* -algebra. In the past, the author worked in characterizations of commutative H^* -algebras (e.g., [3, 4]). As in [3], we do not assume both the existence of an inner product and the commutativity.

Now, we state our first result.

THEOREM 1. *Let A be a Banach algebra with an involution $x \rightarrow x^*$, $x \in A$, such that $\|x^*\| = \|x\|$. Assume that the set $A^2 = \{xy : x, y \in A\}$ has a complex valued trace, that is, there is a complex valued function tr on A^2 with the following properties:*

- (i) *if x, y , and $x + y$ belong to A^2 , then $\text{tr}(x + y) = \text{tr} x + \text{tr} y$,*
- (ii) *$\text{tr}(\lambda x) = \lambda \text{tr} x$ for all $x \in A^2$ and any complex number λ ,*
- (iii) *$\text{tr}(x^*x) \geq 0$ and $\text{tr} x^*x = 0$ if and only if $x = 0$, $x \in A$,*
- (iv) *$\text{tr} x^* = \overline{\text{tr} x}$ for all $x \in A^2$.*

Suppose also that

- (v) *$\text{tr}(xy) = \text{tr}(yx)$ for all $x, y \in A$,*
- (vi) *$|\text{tr}(xy)| \leq \|x\| \cdot \|y\|$ for all $x, y \in A$,*
- (vii) *for each bounded linear functional f ($f \in A^*$), there exists $a \in A$ such that $f(x) = \text{tr}(xa^*)$ for all $x \in A$.*

Then A is a proper H^ -algebra with respect to some scalar product whose corresponding norm $\|\cdot\|$ is equivalent to the original norm. This means that there exists a scalar product $(\cdot, \cdot)$ on A such that A is an H^* -algebra with respect to this scalar product (and the original involution) and such that $k_1\|x\|^2 \leq (x, x) \leq k_2\|x\|^2$ for some $k_1, k_2 \geq 0$ and all $x \in A$ (and $(xy, z) = (y, x^*z) = (x, zy^*)$ for all $x, y, z \in A$).*

REMARK 2. Note that each proper H^* -algebra A has all the properties stated in Theorem 1 [5].

PROOF OF THEOREM 1. For any $x, y \in A$, let $(x, y) = \text{tr}(xy^*) = \text{tr}(y^*x)$. Then $(\cdot, \cdot)$ is an inner product on A [6] (in the terminology of Loomis [1], $(\cdot, \cdot)$ is a scalar product). Let $\|\cdot\|_2$ be the corresponding norm, $(x, x) = \|x\|_2^2$, $x \in A$.

We show that A is complete with respect to this new norm $\|\cdot\|_2$. Let $\{a_n\}$ be a Cauchy sequence, $\lim_{m,n} \|a_n - a_m\|_2 = 0$. Then there exists $M > 0$ such that $\|a_n\|_2 \leq M$ for all n (every Cauchy sequence is bounded). For any fixed $x \in A$, the sequence $\{\text{tr}(xa_n^*)\}$ of complex numbers is also Cauchy

$$(|\text{tr}(xa_n^*) - \text{tr}(xa_m^*)| \leq \|x\|_2 \|a_n - a_m\|_2). \quad (1)$$

So there is a complex number λ_x such that $\text{tr}(xa_n^*) \rightarrow \lambda_x$, $n \rightarrow \infty$. Define the complex valued function f on A by setting $f(x) = \lambda_x$. It follows from

$$f(x) = \lim \text{tr}(xa_n^*), \quad \|a_n\|_2 \leq M, \quad (2)$$

and the linearity of tr that f is a bounded linear functional on A ($f \in A^*$). Assumption (vii) in [Theorem 1](#) implies that there exists $a \in A$ such that

$$(x, a) = f(x) = \lim_n \text{tr} xa_n^*. \quad (3)$$

We show that $\lim_{n \rightarrow \infty} \|a_n - a\|_2 = 0$. Let $\epsilon > 0$ be arbitrary, and let n_0 be such that $\|a_n - a_m\|_2 < \epsilon/2$ for all $n, m > n_0$. Let $n > n_0$ be fixed and arbitrary. The following relation:

$$\begin{aligned} \|a - a_n\|_2^2 &= |(a - a_n, a - a_n)| \\ &= |(a - a_n, a) - (a - a_n, a_n) + (a - a_n, a_n - a_n)| \\ &\leq |f(a - a_n) - (a - a_n, a_n)| + \|a - a_n\|_2 \cdot \|a_n - a_n\|_2 \\ &\leq |f(a - a_n) - (a - a_n, a_n)| + \frac{\epsilon}{2} \|a - a_n\|_2 \end{aligned} \quad (4)$$

shows that $\|a - a_n\|_2^2 \leq \epsilon \|a - a_n\|_2$, since we can always find $m > n_0$ so that $|f(a - a_n) - (a - a_n, a_m)| \leq \epsilon/2 \|a - a_n\|_2$. Hence, $\|a - a_n\|_2 \leq \epsilon$ for any $n > n_0$. This proves that A is complete in this new norm $\|\cdot\|_2$.

It follows from (vi) that $\|x\|_2 \leq \|x\|$ ($(x, x) \leq \|x\| \|x^*\| = \|x\|^2$ for all $x \in A$). Closed graph theorem [\[1\]](#) tells us that $\|\cdot\|_2$ is equivalent to the original norm.

Now, it is an easy exercise to show that A is an H^* -algebra with respect to the inner product $(\cdot, \cdot)$. $\square$

Now we state a slightly different characterization. It may not look like much of improvement over [Theorem 1](#), but it allows for a larger class of examples. In fact, if we take any proper H^* -algebra A and replace its norm by any other norm equivalent to the original one, we get a canonical example of a Banach algebra which both satisfies the conditions of the following theorem and is characterized by it.

THEOREM 3. *Let A be a Banach algebra with continuous involution $x \rightarrow x^*$, $x \in A$. Assume that the set $A^2 = \{xy : x, y \in A\}$ has a trace tr with the following properties:*

- (i) *if $x, y, x + y \in A^2$, then $\text{tr}(x + y) = \text{tr} x + \text{tr} y$,*
- (ii) *$\text{tr}(\lambda x) = \lambda \text{tr} x$ for all $x \in A^2$ and each complex number λ ,*
- (iii) *$\text{tr}(x^* x) \geq 0$ and $\text{tr}(x^* x) = 0$ if and only if $x = 0$ ($x \in A$),*
- (iv) *$\text{tr} x^* = \overline{\text{tr} x}$, $x \in A^2$,*
- (v) *$\text{tr}(xy) = \text{tr}(yx)$ for all $x, y \in A$.*

Assume further that

(vi)' for each $a \in A$ the map $T_a : x \rightarrow \text{tr}(xa^*) (= T_a(x))$ is continuous ($T_a \in A^*$ for each $a \in A$),

(vii) for each bounded linear functional $f (f \in A^*)$ there exists $a \in A$ such that $f(x) = \text{tr}(xa^*)$ for all $x \in A$.

Then A has a structure of a proper H^* -algebra with respect to some scalar product $(,)$ such that $k\|x\|^2 \leq (x, x) \leq K\|x\|^2$ for all $x \in A$ and some $k, K > 0$.

REMARK 4. Note that (vi)' is equivalent to the following condition:

(vi)'' there exists $M > 0$ such that $|\text{tr}(xy)| \leq M\|x\| \cdot \|y\|$ for all $x, y \in A$.

It is a consequence of uniform boundness theorem [6, page 239]. Proof of this fact is similar to the proof of Lemma 1 in [2]. Note also that continuity of involution implies that there exists $B > 0$ such that $\|x^*\| \leq B\|x\|$, $x \in A$.

PROOF OF THEOREM 3. We leave it to the reader to modify the proof of Theorem 1 in order to verify validity of Theorem 3. $\square$

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