ON THE DIOPHANTINE EQUATION \( Ax^2 + 2^{2m} = y^n \)

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Abstract. Let \( h \) denote the class number of the quadratic field \( \mathbb{Q}(\sqrt{-A}) \) for a square free odd integer \( A > 1 \), and suppose that \( n > 2 \) is an odd integer with \( (n, h) = 1 \) and \( m > 1 \). In this paper, it is proved that the equation of the title has no solution in positive integers \( x \) and \( y \) if \( n \) has any prime factor congruent to 1 modulo 4. If \( n \) has no such factor it is proved that there exists at most one solution with \( x \) and \( y \) odd. The case \( n = 3 \) is solved completely. A result of E. Brown for \( A = 3 \) is improved and generalized to the case where \( A \) is a prime \( \not\equiv 7 \pmod{8} \).

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1. Introduction. Let \( A, m, n \) denote positive integers where \( n \) is odd \( > 1 \) and \( A \) square free odd integer. Let \( K = \mathbb{Q}(\sqrt{-A}) \), where \( \mathbb{Q} \) is the field of rational numbers, let further \( h \) denote the number of classes of ideals in \( K \) and suppose \( (h, n) = 1 \). In this paper, we consider the Diophantine equation \( Ax^2 = 2^{2m} = y^n \), where \( x \) and \( y \) are integers. The case \( A = 1 \) was studied in [1] so we will assume that \( A > 1 \). The first result regarding this equation is due to Nagell [5] who proved that when \( m = 0, 1 \), this equation has no solutions in integers \( x \) and \( y \) under the above assumptions about \( A \) and \( n \) so we will suppose that \( m > 0 \). Since \( n \) is odd, there is no loss of generality in considering only odd primes \( p \) and \( x, y \) positive integers, so we will assume this in what follows.

We start by proving the main result of this paper.

Theorem 1.1. Let \( A > 1 \) be a square free odd integer, \( p \) an odd prime with \( (h, p) = 1 \) and \( m \geq 1 \). Then the Diophantine equation

\[ Ax^2 + 2^{2m} = y^n \]

has no solution with \( x \) odd in any of the following cases:

(i) if \( A = 3 \);
(ii) if \( p \equiv 1 \pmod{4} \);
(iii) if \( A \equiv 3 \pmod{4} \) and \( p > 3 \).

For \( p = 3 \), such a solution exists if and only if \( A \) is the square-free part of either \((1/3)(1 + 2^{m+3})\) with \( m \) even or of \((1/3)(2^m - 1)\), although in these cases, there might be other solutions if \( 3 \mid h \).

Proof. We factorize (1.1) in the field \( K \),

\[ \left(2^m + x\sqrt{-A}\right)\left(2^m - x\sqrt{-A}\right) = y^n. \]
Now the principal ideal \([2^m + x\sqrt{-A}]\) and its conjugate ideal are coprime, so \([2^m + x\sqrt{-A}] = \pi p\) for some ideal \(\pi\) in \(K\). It follows that \(\pi p\) is a principal ideal and since \((h, p) = 1\), therefore \(\pi\) is a principal ideal, say \(\pi = [\xi]\) for some element \(\xi\) in \(K\). So we get the equation

\[
[2^m + x\sqrt{-A}] = [\xi]^p, \tag{1.3}
\]

and, consequently,

\[
(2^m + x\sqrt{-A}) = \epsilon \xi^p, \tag{1.4}
\]

for some unit \(\epsilon\) in \(K\). Therefore we have the following three cases:

\[
x\sqrt{-3} + 2^m = \left(\frac{1 \pm \sqrt{-3}}{2}\right) \left(\frac{a + b\sqrt{-3}}{2}\right)^3, \quad a \equiv b \pmod{2},
\]

\[
x\sqrt{-A} + 2^m = \left(\frac{a + b\sqrt{-A}}{2}\right)^p, \quad a \equiv b \equiv 1 \pmod{2}, \tag{1.5}
\]

\[
x\sqrt{-A} + 2^m = \left(\frac{a + b\sqrt{-A}}{2}\right)^p,
\]

for some rational integers \(a\) and \(b\).

Equating the imaginary parts in the first case we get

\[16x = \pm (a^3 - 9ab^2) + (3a^2b - 3b^3), \tag{1.6}\]

and we can absorb the lower sign into \(a\). Then

\[16x = (a + b)^3 - 12ab^2 - 4b^3. \tag{1.7}\]

Since \(a\) and \(b\) have the same parity, we write \(2c = a + b\), and obtain

\[2x = c^3 - 3cb^2 + b^3. \tag{1.8}\]

Equation (1.8) is impossible, since the right-hand side is odd unless both \(b\) and \(c\) are even, and then this side is divisible by 8 if they are which is not possible since \(x\) is odd. So this case does not arise.

The second case arises only if \(A \equiv 3 \pmod{4}\), and we will prove that in this case \(p = 3\) and \(A \neq 3\).

Observe that \(((a + b\sqrt{-A})/2)^3 \in \mathbb{Z}[\sqrt{-A}]\ only if \(A \equiv 3 \pmod{8}\) and \(p = 3\) and then equating the real parts in this case, we get

\[2^{m+3} = a(a^2 - 3Ab^2). \tag{1.9}\]

Since \(a\) is odd we get \(a = \pm 1\) and then

\[\pm 2^{m+3} = 1 - 3Ab^2. \tag{1.10}\]

Now \(A > 1\), so only the negative sign holds and then

\[Ab^2 = \frac{1 + 2^{m+3}}{3}. \tag{1.11}\]
Considering this equation modulo 3 we deduce that \( m \) should be even. If \( A = 3 \), then we get
\[
-2^{m+3} = (1 - 3b)(1 + 3b).
\] (1.12)
So
\[
2^t = 1 + 3b, \quad -2^k = 1 - 3b,
\] (1.13)
where \( t + k = m + 3 \). By adding these two equations we get \( t = 2, k = 1 \), which is impossible since \( m \geq 1 \).

Finally the third case can occur for all \( A \), and we will prove that there is no solution when either \( p \equiv 1 \pmod{4} \) or \( A \equiv 3 \pmod{4} \).

Since \( x \) is odd it follows that \( y = a^2 + Ab^2 \) is odd, so \( a \) and \( b \) have opposite parity. On equating the real parts we get
\[
2^m = a \sum_{r=0}^{(p-1)/2} \binom{p}{2r} a^{p-2r-1} (-Ab^2)^r.
\] (1.14)
Here \( \sum \) is odd, since the first and the last terms have opposite parity and the rest are all even. So \( a = \pm 2^m \), \( b \) is odd and from (1.14) we get
\[
\pm 1 = \sum_{r=0}^{(p-1)/2} \binom{p}{2r} 2^{m(p-2r-1)} (-Ab^2)^r.
\] (1.15)
Then \( \pm 1 \equiv 2^{m(p-1)} \pmod{p} \) and so the lower sign is impossible. That is,
\[
1 = \sum_{r=0}^{(p-1)/2} \binom{p}{2r} 2^{m(p-2r-1)} (-Ab^2)^r,
\] (1.16)
and \( a = 2^m \). So \( y = 2^{2m} + Ab^2 \).

Now suppose that \( p \equiv 1 \pmod{4} \), say \( p = 1 + 2^k u \), where \((u, 2) = 1 \) and \( k \geq 2 \). Since both \( b \) and \( A \) are odd,
\[
b^{p-1} = (bu)^{2k} \equiv 1 \pmod{2^{k+2}}, \quad (-A)^{(p-1)/2} = (Au)^{2^{k-1}} \equiv 1 \pmod{2^{k+1}}.
\] (1.17)
Then from (1.16) we get
\[
1 \equiv \binom{p}{3} 2^{2m} (-Ab^2)^{(p-3)/2} + p b^{p-1} (-A)^{(p-1)/2} \pmod{2^{k+1}}
\] (1.18)
\[
= \frac{p(p-2)}{3} \cdot 2^{k+2m-1} u (-Ab^2)^{(p-1)/3} + p \pmod{2^{k+1}},
\]
since \( m \geq 1 \), therefore \( k + 2m - 1 \geq k + 1 \), so from (1.18) we get \( 3 \equiv 3p \pmod{2^{k+1}} \). Hence \( p \equiv 1 \pmod{2^{k+1}} \) which is not possible. We conclude that there is no solution when \( p \equiv 1 \pmod{4} \). If \( A \equiv 3 \pmod{4} \), then considering (1.16) modulo 4 we get \( p \equiv 1 \pmod{4} \) hence there is no solution when \( A \equiv 3 \pmod{4} \).

Now let \( p = 3 \) in (1.16) then
\[
1 = 2^{2m} - 3Ab^2
\] (1.19)
or \( Ab^2 = (2^{2m} - 1)/3 \). This completes the proof. \( \square \)
Remark 1.2. From Theorem 1.1, we note that to solve (1.1) it is sufficient to consider (1.16) where \( b \) is odd, \( p \equiv 3 \pmod{4} \), and \( A \equiv 1 \pmod{4} \). If there is a solution then \( y = 2^{2m} + Ab^2 \).

Now we prove the following theorem which gives us the number of solutions of our equation.

**Theorem 1.3.** For a given \( A \), if (1.1) has a solution in \( x \) odd where \( (h,p) = 1 \), then it is unique.

**Proof.** If \( A \equiv 3 \pmod{4} \), we have proved that there is a solution only if \( p = 3 \), and we have found this unique solution. If \( A \equiv 1 \pmod{4} \), then from the last proof it is sufficient to consider (1.16), where \( b \) is odd and \( p \equiv 3 \pmod{4} \). Suppose \( b_1 > b > 0 \) is another solution, then from (1.16) we obtain

\[
1 = \sum_{r=0}^{(p-1)/2} \left( \frac{p}{2r} \right) 2^{m(p-2r-1)} (-Ab_1^2)^r. \tag{1.20}
\]

Subtracting (1.20) from (1.16) and dividing by \( b_1^2 - b^2 \), we get

\[
0 = \sum_{r=0}^{(p-1)/2} \left( \frac{p}{2r} \right) \frac{b_1^{2r} - b^{2r}}{b_1^2 - b^2} \cdot 2^{m(p-2r-1)} (-A)^r \tag{1.21}
\]

Since \( p \equiv 3 \pmod{4} \), the number \( (b_1^{p-1} - b^{p-1})/(b_1^2 - b^2) \) is odd, so (1.21) is impossible and the solution is unique as required.

Now we prove that to solve (1.1) it is sufficient to consider only \( x \) odd. First we need the following lemma.

**Lemma 1.4 [4].** The Diophantine equations

\[
Ax^2 + 1 = 2^{2n}, \quad A \equiv 1 \pmod{4}, \quad Ax^2 + 1 = 4^{\nu n}, \quad A \equiv 3 \pmod{4}, \tag{1.22}
\]

have no solutions in positive integers with \( y > 1, n > 2, 2 \nmid ny \) and \( (n,h) = 1 \).

**Theorem 1.5.** If \( A = 3 \), equation (1.1) has a solution with \( x \) even only if \( m \equiv -1 \pmod{p} \), and this solution is given by \( x = 2^m \); for all other \( A \equiv 7 \pmod{8} \) with \( (h,p) = 1 \) there exists a solution with \( x \) even of the form \( x = 2^nX \) with \( X \) odd, if and only if there is a solution of the equation \( AX^2 + 2^{2(m-1)} = Yp \).

**Proof.** If \( x \) is even then \( y \) is even, so let \( x = 2^uX, \; y = 2^v \cdot Y \), where \( u > 0, \; v > 0, \; (2,X) = (2,Y) = 1 \). Then (1.1) becomes

\[
A(2^uX)^2 + 2^{2m} = 2^{vp}Yp. \tag{1.23}
\]

We have three cases:

1. \( pv > 2u = 2m \). Then cancelling \( 2^{2m} \) in (1.23) we get

\[
AX^2 + 1 = 2^{v(p-2m)}Yp, \tag{1.24}
\]

where \( X \) is odd. Now \( A \equiv 7 \pmod{8} \), so \( v \cdot p - 2m = 1 \) or 2.
If \( A \equiv 1 \pmod{4} \) then \( yp - 2m = 1 \) and so \( AX^2 + 1 = 2YP \). This equation has no solution from Lemma 1.4. If \( A \equiv 3 \pmod{8} \), then \( yp - 2m = 2 \), so \( AX^2 + 1 = 4YP \), and again from Lemma 1.4 this equation has no solution in integers with \( Y > 1 \). Let \( Y = 1 \), then \( AX^2 + 1 = 4YP \) implies that \( A = 3 \), \( X = 1 \) and hence \( x = 2^m \), also \( yp = 2m + 2 \) implies that \( m \equiv -1 \pmod{p} \).

(2) \( 2u > 2m = vp \). Then canceling \( 2^2m \) in (1.23) we get \( A(2^u - mX)^2 + 1 = Yp \). This equation has no solution [5, Theorem 25].

(3) \( 2m > 2u = pv \). Then

\[
AX^2 + 2^{2(m-u)} = Yp, 
\]

and this is (1.1) with \( x \) odd and smaller \( m \).

\[ \square \]

**Remark 1.6.** From the proof of the last theorem we deduce that to solve (1.1) in even integers when \( A \neq 3 \) and \( A \neq 7 \pmod{8} \), it is sufficient to consider the equation

\[
AX^2 + 2^{2(m-u)} = Yp, 
\]

where \( x = 2uX, y = 2^vY, m > u > 0, v > 0, (2, X) = (2, Y) = 1, \) and \( 2u = pv \).

Summarizing the above we give the following theorem.

**Theorem 1.7.** The Diophantine equation (1.1) where \( A \neq 7 \pmod{8} \) and \( (h, p) = 1 \) has no integer solution if \( p \equiv 1 \pmod{4} \). In particular, the equation \( px^2 + 2^{2m} = y^n \) has no solution for all \( p > 3 \) and \( p \neq 7 \pmod{8} \).

**Proof.** If \( x \) is odd, then from Theorem 1.1, equation (1.1) has no solution when \( p \equiv 1 \pmod{4} \). Now let \( x \) be even then from Theorem 1.5 it is sufficient to consider the equation

\[
AX^2 + 2^{2(m-u)} = Yp, 
\]

where \( X \) is odd and \( 0 < u < m \). Since \( p \equiv 1 \pmod{4} \) then again Theorem 1.1 implies that there is no solution.

Now the class number of the field \( \mathbb{Q}(\sqrt{-p}) \) is less than \( p \), so as above the equation \( px^2 + 2^{2m} = y^n \) has no solution if \( p \equiv 1 \pmod{4} \). Let \( p \equiv 3 \pmod{4} \), since \( p > 3 \), therefore the equation has no solution in odd integers from Theorem 1.1(iii). If \( x \) is even then we have

\[
px^2 + 2^{2(m-u)} = Yp, 
\]

where \( X \) is odd. Equation (1.28) has no solution in odd integers from the first part.

\[ \square \]

Brown [2, Theorem 3] considered the Diophantine equation (1.1) when \( A = 3 \), but he did not solve it completely. In the following we give the complete solution.

**Theorem 1.8.** The Diophantine equation \( 3x^2 + 2^{2m} = y^n \) has a solution only if \( m \equiv -1 \pmod{p} \), and this solution is given by \( x = 2^m, y = 2^{(2m+2)/p} \).

**Proof.** Now \( A = 3 \) and the field \( \mathbb{Q}(\sqrt{-3}) \) is a unique prime factorization domain, so from Theorem 1.1 this equation has no solution for all \( p \) if \( x \) is odd. If \( x \) is even then from Theorem 1.5 we have \( x = 2^m, y = 2^{(2m+2)/p} \). Also the equation

\[
3X^2 + 2^{2(m-u)} = Yp, 
\]

where \( X \) is odd, has no solution from the first part of this proof.
Combining the two last theorems we can generalize Brown’s result [2] for any odd prime \( p \) as follows.

**Theorem 1.9.** The Diophantine equation \( px^2 + 2^{2m} = y^n \), where \( p \not\equiv 7 \pmod{8} \), has a solution only if \( p = 3 \) and \( m \equiv 2 \pmod{3} \) and this solution is given by \( x = 2^m \), \( y = 2(2m + 2)/3 \).

Considering (1.16) modulo 8 it is easy to prove the following.

**Corollary 1.10.** For a given \( A \), in (1.16) where \( m \geq 2 \), if \( A \equiv 1 \pmod{8} \) then \( p \equiv 7 \pmod{8} \) and if \( A \equiv 5 \pmod{8} \) then \( p \equiv 3 \pmod{8} \).

As a special case we consider \( A = q \) an odd prime and prove the following theorem.

**Theorem 1.11.** The Diophantine equation \( qx^2 + 2^{2m} = y^3 \), where \( q \equiv 1 \pmod{4} \) is a prime integer and \((3, h) = 1\), has a solution only if \( q = 5 \) and \( m = 2 + 3M \), and the unique solution is given by \( x = 43 \cdot 2^{3M} \) and \( y = 21 \cdot 2^{2M} \).

**Proof.** First suppose that \( x \) is odd, since \( q \equiv 1 \pmod{4} \) and \( p = 3 \), therefore it is sufficient to consider (1.16), then \( y = 2^{2m} + qb^2 \) and

\[
3qb^2 = 2^{2m} - 1 = (2^m - 1)(2^m + 1). \tag{1.30}
\]

From [5] it is sufficient to consider \( m \geq 2 \) and from Corollary 1.10 we have \( q \equiv 5 \pmod{8} \). Now \((2m - 1, 2^m + 1) = 1\), let \( b = cd \), where \((c, d) = 1\) and both \( c \) and \( d \) are odd, then from (1.30) we have only the following possibilities:

1. \( 2^m - 1 = 3qc^2, 2^m + 1 = d^2 \), subtracting these two equations, we get \( 2 = d^2 - 3qc^2 \) which is not possible modulo 3.
2. \( 2^m - 1 = 3c^2, 2^m + 1 = qd^2 \), considering the first equation modulo 8, we get \( m = 2 \) and hence \( q = 5 \). Therefore \( y = 2^{2m} + qb^2 = 2^4 + 5(1) = 21 \) and so \( x = 43 \).
3. \( 2^m - 1 = d^2, 2^m + 1 = 3qc^2 \), again considering the first equation modulo 8, we get \( m = 1 \) and then \( q = 1 \) which is not our case.
4. \( 2^m - 1 = qc^2, 2^m + 1 = 3d^2 \), considering the first equation modulo 8, we get a contradiction.

Now suppose that \( x \) is even, then we have only the following equation:

\[
qX^2 + 2^{2(m-u)} = Y^3, \tag{1.31}
\]

where \( x = 2^uX, \ y = 2^vY, \ m > u > 0, \ v > 0, (2, X) = (2, Y) = 1, \) and \( 2u = 3v \). From the first part of this proof, equation (1.31) has a unique solution given by \( q = 5, m - u = 2, X = 43, \) and \( Y = 21 \). Since \( 2u = 3v \) we get \( 3 \mid u \), let \( u = 3M \) then \( m = 2 + 3M \) and \( v = 2M \). Hence \( x = 43 \cdot 2^{3M} \) and \( y = 21 \cdot 2^{2M} \).

We are unable to solve (1.1) completely when \( A \equiv 1 \pmod{4} \) but we are able to solve it for many particular values of \( A \) for all \( p \) as we will show in the following example. But before this we give a corollary which will help us.

**Corollary 1.12.** If \( m \) is odd then the Diophantine equation (1.1) has no solution in \( x \) odd when \( 5 \mid A \).
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**Proof.** Since \( m \) is odd, therefore from the proof of **Theorem 1.1**, it is sufficient to consider (1.16), where \( p = 3 \pmod{4} \). If \( 5 \mid A \) in (1.16), then we get \( 1 = 2^{m(p-1)} \pmod{5} \) which implies that \( 4 \mid m(p-1) \) and this is not possible.

**Example 1.13.** Consider the Diophantine equation \( 5x^2 + 2^{10} = y^p \).

Here \( m = 5, A = 5, h = 2 \), so from **Corollary 1.12**, this equation has no solution in \( x \) odd for all \( p \). If \( x \) is even then it is sufficient to consider the equation

\[
5X^2 + 2^{5(u-1)} = Y^p,
\]

(1.32)

where \( x = 2^uX, y = 2^vY, 5 > u > 0, v > 0, (2, X) = (2, Y) = 1 \), and \( 2u = pv \). Since \( p \) is an odd prime, the only possibility is \( u = 3, p = 3 \), and (1.32) becomes \( 5X^2 + 2^3 = Y^3 \), which has a unique solution from **Theorem 1.11**, given by \( X = 43 \) and \( Y = 21 \), so the given equation has a unique solution, \( x = 8.43, y = 4.21 \), and \( p = 3 \).

By using the method similar to [3, Lemma 3] we can prove the following lemma.

**Lemma 1.14.** If \( q \) is any odd prime which divides the integer \( b \) defined in (1.16), then

\[
2^{m(q-1)/2} \equiv 1 \pmod{q^2}.
\]

(1.33)

Considering (1.16) modulo 3 we are able to prove the following theorem.

**Theorem 1.15.** If \( 3 \mid b \) in (1.16), then \( m = 3^k \cdot m' \), where \( k \geq 1 \), \( (3, m') = 1 \) and either

1. \( 3 \mid A \) and then there is no solution if \( k \) even, or
2. \( 3 \mid A \) and then there is no solution if \( k \) odd.

**Proof.** Let \( 3 \mid b \) then from **Lemma 1.14**, \( 2^{2m} \equiv 1 \pmod{9} \) which implies that \( 3 \mid m \).

Let \( m = 3^k \cdot m' \), where \( (3, m') = 1 \), \( k \geq 1 \). Since \( p = 3 \pmod{4} \), put \( p-1 = 2 \cdot 3^l \cdot p' \), where \( (2, p') = (3, p') = 1 \), \( t \geq 0 \) and put \( b = 3^s \cdot b' \), where \( (3, b') = 1, s \geq 1 \). Rewrite (1.16) as

\[
1 - 2^{m(p-1)} = \sum_{r=1}^{(p-1)/2} \left( \frac{p}{2r} \right) 2^{m(p-2r-1)} \cdot (-Ab^2)^r.
\]

(1.34)

The general term in the right-hand side is

\[
\left( \frac{p}{2r} \right) 2^{m(p-2r-1)} \cdot (-Ab^2)^r = \left( \frac{p-2}{2r-2} \right) 2^{m(p-2r-1)} \times \frac{pb^{2r-2}}{r(2r-1)} \cdot b^2(-A)^r.
\]

(1.35)

Since \( 3^{2r-2} \geq r(2r-1) \) for \( r \geq 1 \), this right-hand side is divisible at least by \( 3^{2s+t} \) if \( (3, A) = 1 \), so from (1.34) we get

\[
2^{m(p-1)} \equiv 1 \pmod{3^{2s+t}}.
\]

(1.36)

Since \( 2 \) is a primitive root of \( 3^{2s+t} \), therefore \( \phi(3^{2s+t}) \mid m(p-1) \) which implies that \( 3^{2s+t-1} \mid 2 \cdot 3^k m' \cdot 3^l p' \), hence \( 3^{2s-k-1} \mid m' p' \). But \( (3, m') = (3, p') = 1 \), so \( 2s - k - 1 = 0 \) which implies that \( k \) is odd.

Now if \( 3 \mid A \), then the right-hand side in (1.34) is divisible at least by \( 3^{2s+t+1} \) and as above we get \( k = 2s \), implying \( k \) even.

\( \square \)
We are unable to solve (1.1) completely when \( p = 7 \), but as a special case we prove the following theorem.

**Theorem 1.16.** The Diophantine equation (1.1), where \((7, h) = 1\), has no solution in \( x \) odd when \( p = 7 \), \( A \equiv 1 \) (mod \( 12 \)), and \( m = 3^k \cdot m' \), where \( k \geq 1 \), \((3, m') = 1\).

**Proof.** Here \( p = 7 \), so from Theorem 1.1(iii) we get \( A \equiv 1 \) (mod \( 4 \)). Put \( p = 7 \) in (1.16), then
\[
1 = 2^{6m} - 21A b^2 2^{4m} + 35A^2 b^4 2^{2m} - 7A^3 b^6, \tag{1.37}
\]
If \( 3 \mid b \) in (1.37) then from Theorem 1.15(1), this equation has no solution. So \( (3, b) = 1 \), and then considering (1.37) modulo 3 we get
\[
2A^2 - A^3 \equiv 0 \pmod{3} \tag{1.38}
\]
which is not true since \( A \equiv 1 \) (mod \( 3 \)). \( \square \)

**Theorem 1.17.** If \((3, m) = 1\), then the Diophantine equation (1.1), where \((p, h) = 1\) has no solution in \( x \) odd when \( A \equiv 1 \) (mod \( 24 \)).

**Proof.** The case \( m = 1 \), the Diophantine equation (1.1) has no solution [5]. Let \( m \geq 2 \), since \( A \equiv 1 \) (mod \( 8 \)) then from Corollary 1.10, \( p = 7 + 8H \). Since \((3, m) = 1\), then from Theorem 1.15, \((3, b) = 1\) so \( b^2 \equiv 1 \) (mod \( 3 \)). Considering (1.16) modulo 3, where \( A \equiv 1 \) (mod \( 3 \)) we get
\[
1 \equiv \sum_{r=0}^{(p-1)/2} \left( \frac{p}{2r} \right) (-1)^r \pmod{3} \equiv \frac{(1+i)^p + (1-i)^p}{2} \pmod{3}. \tag{1.39}
\]
But \((1 \pm i)^8 \equiv 1 \) (mod \( 3 \)), so (1.39) implies that
\[
1 \equiv \frac{(1+i)^8(1+i)(1-i) + (1-i)^8(1-i)(1+i)}{2(1+i)(1-i)} \equiv 4^{1+H} \times \frac{1}{2} \pmod{3} \tag{1.40}
\]
\[
\equiv 2 \pmod{3}
\]
which is a contradiction. \( \square \)

**Example 1.18.** Consider the Diophantine equation \( 73x^2 + 2^{14} = y^p \).

Here \( m = 7 \), \( A = 73 \), \( h = 4 \) so from Theorem 1.17, this equation has no solution in \( x \) odd. If \( x \) is even then it is sufficient to consider the equation
\[
73X^2 + 2^{2(u/2)} = Y^p, \tag{1.41}
\]
where \( x = 2^u X \), \( y = 2^v \cdot Y \), \( 7 > u > 0 \), \( v > 0 \), \((2, X) = (2, Y) = 1\), and \( 2u = pv \). If \((7-u, 3) = 1\), then (1.41) has no solution from Theorem 1.17. If \( 3 \mid 7-u \) then \( u = 1 \) or 4, which is not possible since \( 2u = pv \). So the given equation has no solution in integers.

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