A GENERALIZATION THEOREM COVERING MANY ABSOLUTE SUMMABILITY METHODS

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ABSTRACT. A general theorem concerning many absolute summability methods is proved.

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1. Introduction. Let $\sum a_n$ be given infinite series with the sequence of partial sums (s_n) . By σ_n^δ and μ_n^δ we denote n th Cesaro mean of order δ ($\delta > -1$) of the sequences (s_n) and (na_n) , respectively. The series $\sum a_n$ is said to be summable $|C, \delta|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\delta - \sigma_{n-1}^\delta|^k < \infty, \quad (1.1)$$

or equivalently

$$\sum_{n=1}^{\infty} n^{-1} |\mu_n^\delta|^k < \infty. \quad (1.2)$$

Let (p_n) be a sequence of real or complex numbers such that

$$P_n = \sum_{\nu=0}^n p_\nu \rightarrow \infty \quad \text{as } n \rightarrow \infty \quad (P_{-i} = p_{-i} = 0, i \geq 1). \quad (1.3)$$

The sequence-to-sequence transformation:

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^n p_\nu s_\nu, \quad (1.4)$$

defines the sequence (t_n) of the Riesz means of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [4]). The series $\sum a_n$ is said to be summable $|R, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty. \quad (1.5)$$

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$ (see [1]), if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |t_n - t_{n-1}|^k < \infty. \quad (1.6)$$

In the special case when $p_n = 1$ for all values of n (respectively, $k = 1$), then both of $|R, p_n|_k$ and $|\bar{N}, p_n|_k$ is the same as $|C, 1|_k$ (respectively, $|R, p_n|$, $|\bar{N}, p_n|$) summability.

For $p_n = 1/(n+1)$, the summability $|\bar{N}, p_n|_k$ is equivalent to $|R, \log n, 1|_k$. The series $\sum a_n$ is said to be summable $|N, p_n|$, if

$$\sum_{n=1}^{\infty} |T_n - T_{n-1}| < \infty, \quad (1.7)$$

where

$$T_n = \frac{1}{P_n} \sum_{v=0}^n p_{n-v} s_v \quad (T_{-1} = 0). \quad (1.8)$$

We write $p = \{p_n\}$ and

$$M = \left\{ p : p_n > 0 \text{ and } \frac{p_{n+1}}{p_n} \leq \frac{p_{n+2}}{p_{n+1}} \leq 1, n = 0, 1, \dots \right\}. \quad (1.9)$$

It is known that for $p \in M$, equation (1.7) holds if and only if (see [3])

$$\sum_{n=1}^{\infty} \frac{1}{nP_n} \left| \sum_{v=1}^n p_{n-v} v a_v \right| < \infty. \quad (1.10)$$

For $p \in M$, the series $\sum a_n$ is said to be summable $|N, p_n|_k, k \geq 1$ (see [5]), if

$$\sum_{n=1}^{\infty} \frac{1}{nP_n^k} \left| \sum_{v=1}^n p_{n-v} v a_v \right|^k < \infty. \quad (1.11)$$

In the special case in which $p_n = A_n^{r-1}, r > -1$, where A_n^r is the coefficient of x^n in the power series of $(1-x)^{-r-1}$, for $|x| < 1$, $|N, p_n|_k$ summability reduces to $|C, r|_k$ summability, and for $p_n = 1/(n+1)$, $|N, p_n|_k$ is equivalent to $|N, 1/(n+1)|_k$ summability. We assume that $(f_n), (g_n), (G_n)$, and (H_n) are positive sequences.

2. Main result. We prove the following theorem.

THEOREM 2.1. *Let*

$$Y_n = \frac{1}{F_{n-1}H_n} \sum_{v=1}^n v f_{n-v} x_v \epsilon_v \quad (2.1)$$

such that

$$F_n = \sum_{v=1}^n f_v \rightarrow \infty, \quad f \in M, \quad X_n = \frac{1}{G_n} \sum_{v=1}^n g_v x_v. \quad (2.2)$$

Suppose that $\{nG_n\epsilon_n/F_n g_n H_n\}$ is monotonic and

$$g_{n+1} = O(g_n), \quad (2.3)$$

$$\Delta g_n = O\left(\frac{g_{n+1}}{n}\right), \quad (2.4)$$

$$\Delta\left(\frac{g_n H_n}{n G_n}\right) = O\left(\frac{g_n H_n}{n G_n F_{n+1}}\right), \quad (2.5)$$

$$\sum_{n=v+1}^m \frac{f_{n-v}}{F_{n-1} H_n^k} = O\left(\frac{1}{H_v^k}\right). \quad (2.6)$$

Then necessary and sufficient conditions that $\sum |Y_n|^k < \infty$ whenever $\sum |X_n|^k < \infty$ are

- (i) $\epsilon_n = O(F_n g_n H_n / n G_n)$,
- (ii) $\Delta \epsilon_n = O(g_{n+1} H_n / n G_n)$.

3. Lemmas

LEMMA 3.1 (see [2]). Let $k \geq 1$, and let $A = (a_{nv})$ be an infinite matrix. In order that $A \in (\ell^k; \ell^k)$ it is necessary that

$$a_{nv} = O(1) \quad (\text{all } n, v). \quad (3.1)$$

LEMMA 3.2 (see [6]). Let $p \in M$. Then for $0 < r \leq 1$,

$$\sum_{n=v+1}^{\infty} \frac{p_{n-v-1}}{n^r p_{n-1}} = O(v^{-r}). \quad (3.2)$$

LEMMA 3.3. Suppose that $\epsilon_n = O(f_n g_n)$, $f_n, g_n \geq 0$, $\{\epsilon_n / f_n g_n\}$ monotonic, $\Delta g_n = O(1)$, and $\Delta f_n = O(f_n / g_{n+1})$. Then $\Delta \epsilon_n = O(f_n)$.

PROOF. Let $k_n = \epsilon_n / f_n g_n = O(1)$. If (k_n) is nondecreasing, then

$$\begin{aligned} \Delta \epsilon_n &= k_n f_n g_n - k_{n+1} f_{n+1} g_{n+1} \leq k_n f_n g_n - k_n f_{n+1} g_{n+1} \\ &= k_n (\Delta f_n g_n) = k_n (f_n \Delta g_n + g_{n+1} \Delta f_n), \\ |\Delta \epsilon_n| &= O(f_n |\Delta g_n|) + O(g_{n+1} |\Delta f_n|) = O(f_n) + O(f_n) = O(f_n). \end{aligned} \quad (3.3)$$

If (k_n) is nonincreasing, write $\nabla f_n = f_{n+1} - f_n$,

$$\begin{aligned} \nabla \epsilon_n &= k_{n+1} f_{n+1} g_{n+1} - k_n f_n g_n \leq k_n \nabla (f_n g_n) \\ &= k_n (f_n \nabla g_n + g_{n+1} \nabla f_n), \\ |\Delta \epsilon_n| &= |\nabla \epsilon_n| = O(f_n |\nabla g_n|) + O(g_{n+1} |\nabla f_n|) \\ &= O(f_n |\nabla g_n|) + O(g_{n+1} |\Delta f_n|) \\ &= O(f_n) + O(f_n) = O(f_n). \end{aligned} \quad (3.4)$$

□

4. Proof of Theorem 2.1

SUFFICIENCY. We have via Abel's transformation:

$$\begin{aligned} Y_n &= \frac{1}{F_{n-1} H_n} \sum_{v=1}^n g_v x_v \left(v \frac{f_{n-v}}{g_v} \epsilon_v \right) \\ &= \frac{1}{F_{n-1} H_n} \left[\sum_{v=1}^{n-1} \left(\sum_{r=1}^v g_r x_r \right) \Delta v \left(\frac{f_{n-v}}{g_v} \epsilon_v \right) + \left(\sum_{r=1}^n g_r x_r \right) n \frac{f_0}{g_n} \epsilon_n \right] \\ &= \frac{1}{F_{n-1} H_n} \sum_{v=1}^{n-1} G_v X_v \left\{ -\frac{f_{n-v}}{g_v} \epsilon_v + (v+1) \Delta g_v^{-1} f_{n-v} \epsilon_v + (v+1) g_{v+1}^{-1} \Delta v f_{n-v} \epsilon_v \right. \\ &\quad \left. + (v+1) g_{v+1}^{-1} f_{n-v-1} \Delta \epsilon_v \right\} + \frac{n G_n X_n f_0}{F_{n-1} H_n g_n} \epsilon_n \\ &= Y_{n,1} + Y_{n,2} + Y_{n,3} + Y_{n,4} + Y_{n,5}. \end{aligned} \quad (4.1)$$

By Minkowski's inequality,

$$\sum_{n=1}^m |Y_n|^k = O(1) \sum_{n=1}^m \sum_{r=1}^5 |Y_{n,r}|^k. \quad (4.2)$$

Applying Hölder's inequality,

$$\begin{aligned} \sum_{n=2}^{m+1} |Y_{n,1}|^k &= \sum_{n=2}^{m+1} \frac{1}{F_{n-1}^k H_n^k} \left| \sum_{v=1}^{n-1} f_{n-v} \frac{G_v}{g_v} X_v \epsilon_v \right|^k \\ &\leq \sum_{n=2}^{m+1} \frac{1}{F_{n-1}^k H_n^k} \sum_{v=1}^{n-1} f_{n-v} \left(\frac{G_v}{g_v} \right)^k |X_v|^k |\epsilon_v|^k \left\{ \sum_{v=1}^{n-1} \frac{f_{n-v}}{F_{n-1}} \right\}^{k-1} \\ &\leq O(1) \sum_{v=1}^m \left(\frac{G_v}{g_v} \right)^k |X_v|^k |\epsilon_v|^k \sum_{n=v+1}^{m+1} \frac{f_{n-v}}{F_{n-1} H_n^k} \\ &\leq O(1) \sum_{v=1}^m \frac{1}{H_v^k} \left(\frac{v}{F_{v-1}} \right)^k \left(\frac{G_v}{g_v} \right)^k |X_v|^k |\epsilon_v|^k, \\ \sum_{n=2}^{m+1} |Y_{n,2}|^k &= \sum_{n=2}^{m+1} \frac{1}{F_{n-1}^k H_n^k} \left| \sum_{v=1}^{n-1} (v+1) G_v \Delta g_v^{-1} f_{n-v} X_v \epsilon_v \right|^k \\ &\leq O(1) \sum_{n=2}^{m+1} \frac{1}{F_{n-1}^k H_n^k} \sum_{v=1}^{n-1} v^k G_v^k |\Delta g_v^{-1}|^k f_{n-v} |X_v|^k |\epsilon_v|^k \left\{ \sum_{v=1}^{n-1} \frac{f_{n-v}}{F_{n-1}} \right\}^{k-1} \\ &\leq O(1) \sum_{v=1}^m v^k G_v^k |\Delta g_v^{-1}|^k |X_v|^k |\epsilon_v|^k \sum_{n=v+1}^{m+1} \frac{f_{n-v}}{F_{n-1} H_n^k} \\ &= O(1) \sum_{v=1}^m \left(\frac{v}{H_v} \right)^k G_v^k \frac{|\Delta g_v|}{g_v^k g_{v+1}^k} |X_v|^k |\epsilon_v|^k \\ &\leq O(1) \sum_{v=1}^m \frac{1}{H_v^k} \left(\frac{v}{F_{v-1}} \right)^k \left(\frac{G_v}{g_v} \right)^k |X_v|^k |\epsilon_v|^k, \\ \sum_{n=2}^{m+1} |Y_{n,3}|^k &= \sum_{n=2}^{m+1} \frac{1}{F_{n-1}^k H_n^k} \left| \sum_{v=1}^{n-1} v g_{v+1}^{-1} G_v \Delta_v f_{n-v} X_v \epsilon_v \right|^k \\ &\leq \sum_{n=2}^{m+1} \frac{1}{F_{n-1}^k H_n^k} \sum_{v=1}^{n-1} v^k \left(\frac{G_v}{g_{v+1}} \right)^k |\Delta_v f_{n-v}| |X_v|^k |\epsilon_v|^k \left\{ \sum_{v=1}^{n-1} |\Delta f_{n-v}| \right\}^{k-1} \\ &\leq O(1) \sum_{v=1}^m v^k \left(\frac{G_v}{g_v} \right)^k |X_v|^k |\epsilon_v|^k \sum_{n=v+1}^{m+1} \frac{\Delta_v f_{n-v}}{F_{n-1}^k H_n^k} \\ &\leq O(1) \sum_{v=1}^m \frac{1}{H_v^k} \left(\frac{v}{F_{v-1}} \right)^k \left(\frac{G_v}{g_v} \right)^k |X_v|^k |\epsilon_v|^k, \end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{m+1} |Y_{n,4}|^k &= \sum_{n=2}^{m+1} \frac{1}{F_{n-1}^k H_n^k} \left| \sum_{v=1}^{n-1} v g_{v+1}^{-1} f_{n-v-1} G_v X_v \Delta \epsilon_v \right|^k \\
&\leq \sum_{n=2}^{m+1} \frac{1}{F_{n-1}^k H_n^k} \sum_{v=1}^{n-1} v^k \left(\frac{G_v}{g_{v+1}} \right)^k f_{n-v-1} |X_v|^k |\Delta \epsilon_v|^k \left\{ \sum_{v=1}^{n-1} \frac{f_{n-v-1}}{F_{n-1}} \right\}^{k-1} \\
&\leq O(1) \sum_{v=1}^m v^k \left(\frac{G_v}{g_{v+1}} \right)^k |X_v|^k |\Delta \epsilon_v|^k \sum_{n=v+1}^{m+1} \frac{f_{n-v-1}}{F_{n-1} H_n^k} \\
&\leq O(1) \sum_{v=1}^m \left(\frac{v}{H_v} \right)^k \left(\frac{G_v}{g_{v+1}} \right)^k |X_v|^k |\Delta \epsilon_v|^k, \\
\sum_{n=1}^m |Y_{n,5}|^k &= \sum_{n=1}^m \left| \frac{n G_n X_n f_0 \epsilon_n}{F_{n-1} H_n g_n} \right|^k \\
&\leq O(1) \sum_{n=1}^m \left(\frac{n}{F_{n-1}} \right)^k \left(\frac{G_n}{g_n} \right)^k \frac{1}{H_n^k} |X_n|^k |\epsilon_n|^k.
\end{aligned} \tag{4.3}$$

Sufficiency of (i) and (ii) follows. Necessity of (i): using the result of Bor in [2], the transformation from (X_n) into (Y_n) maps ℓ^k into ℓ^k and hence the diagonal elements of this transformation are bounded (by Lemma 3.1) and so (i) is necessary. Necessity of (ii): this follows from Lemma 3.3 and necessity of (i) by taking

$$f_n \equiv \frac{g_n H_n}{n G_n}, \quad g_n \equiv F_n \text{ using (2.3).} \tag{4.4}$$

This completes the proof of the theorem.

REMARKS. It may be mentioned that on putting

- (1) $f_n = p_n$ and $H_n = n^{1/k}$, we obtain $|N, p_n|_k$ summability of $\sum a_n \epsilon_n$.
- (2) $g_n = Q_{n-1}$ and $G_n = Q_{n-1} (Q_n/q_n)^{1/k}$, we obtain $|\bar{N}, q_n|_k$ summability of $\sum a_n$.
- (3) $g_n = Q_{n-1}$ and $G_n = n^{1/k-1} (Q_n Q_{n-1}/q_n)$, we obtain $|R, q_n|_k$ summability of $\sum a_n$.

5. Applications

THEOREM 5.1. Let $p \in M$, $\{(n \epsilon_n / P_n)(Q_n / n q_n)^{1/k}\}$ is monotonic,

$$\Delta \left(\frac{1}{n} \left(\frac{n q_n}{Q_n} \right)^{1/k} \right) = O \left(\frac{1}{n} \left(\frac{n q_n}{Q_n} \right)^{1/k} \frac{1}{P_{n+1}} \right), \quad n q_n = O(Q_n). \tag{5.1}$$

Then necessary and sufficient conditions that $\sum a_n \epsilon_n$ be summable $|N, p_n|_k$ whenever $\sum a_n$ is summable $|\bar{N}, q_n|_k$, $k \geq 1$, are

$$\epsilon_n = O \left\{ \frac{P_n}{n} \left(\frac{n q_n}{Q_n} \right)^{1/k} \right\}, \quad \Delta \epsilon_n = O \left\{ \frac{1}{n} \left(\frac{n q_n}{Q_n} \right)^{1/k} \right\}. \tag{5.2}$$

THEOREM 5.2. Let $p \in M$, $\{Q_n \epsilon_n / P_n q_n\}$ is monotonic,

$$\Delta \left(\frac{q_n}{Q_n} \right) = O \left(\frac{q_n}{Q_n P_{n+1}} \right), \quad n q_n = O(Q_n). \tag{5.3}$$

Then necessary and sufficient conditions that $\sum a_n \epsilon_n$ be summable $|N, p_n|_k$ whenever $\sum a_n$ is summable $|R, q_n|_k$, $k \geq 1$ are

$$\epsilon_n = O\left(\frac{P_n q_n}{Q_n}\right), \quad \Delta \epsilon_n = O\left(\frac{q_n}{Q_n}\right). \quad (5.4)$$

COROLLARY 5.3. *Necessary and sufficient conditions that $\sum a_n \epsilon_n$ be summable $|C, \alpha|_k$, $0 < \alpha \leq 1$ whenever $\sum a_n$ is summable $|C, 1|_k$, $k \geq 1$, are*

$$\epsilon_n = O(n^{\alpha-1}), \quad \Delta \epsilon_n = O(n^{-1}), \quad (5.5)$$

provided $\{n^{1-\alpha} \epsilon_n\}$ is monotonic.

COROLLARY 5.4. *Necessary and sufficient conditions that $\sum a_n \epsilon_n$ be summable $|N, 1/(n+1)|_k$ whenever $\sum a_n$ is summable $|C, 1|_k$, $k \geq 1$, are*

$$\epsilon_n = O\left(\frac{\log n}{n}\right), \quad \Delta \epsilon_n = O(n^{-1}), \quad (5.6)$$

provided $\{n \epsilon_n / \log n\}$ is monotonic.

COROLLARY 5.5. *Necessary and sufficient conditions that $\sum a_n \epsilon_n$ be summable $|N, 1/(n+1)|_k$ whenever $\sum a_n$ is summable $|R, \log n, 1|_k$, $k \geq 1$, are*

$$\epsilon_n = O\left\{\frac{(\log n)^{1-1/k}}{n}\right\}, \quad \Delta \epsilon_n = O\left\{\frac{1}{n(\log n)^{1/k}}\right\}, \quad (5.7)$$

provided $\{n(\log n)^{1/k-1} \epsilon_n\}$ is monotonic.

COROLLARY 5.6. *Necessary and sufficient conditions that $\sum a_n \epsilon_n$ be summable $|C, \alpha|_k$, $0 < \alpha \leq 1$ whenever $\sum a_n$ is summable $|R, \log n, 1|_k$, $k \geq 1$, are:*

$$\epsilon_n = O\left\{\frac{n^{\alpha-1}}{(\log n)^{1/k}}\right\}, \quad \Delta \epsilon_n = O\left\{\frac{1}{n(\log n)^{1/k}}\right\}, \quad (5.8)$$

provided $\{n^{1-\alpha}(\log n)^{1/k} \epsilon_n\}$ is monotonic.

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