NEAR FRATTINI SUBGROUPS OF RESIDUALLY FINITE GENERALIZED FREE PRODUCTS OF GROUPS

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ABSTRACT. Let $G = A \star_H B$ be the generalized free product of the groups A and B with the amalgamated subgroup H . Also, let $\lambda(G)$ and $\psi(G)$ represent the lower near Frattini subgroup and the near Frattini subgroup of G , respectively. If G is finitely generated and residually finite, then we show that $\psi(G) \leq H$, provided H satisfies a nontrivial identical relation. Also, we prove that if G is residually finite, then $\lambda(G) \leq H$, provided: (i) H satisfies a nontrivial identical relation and A, B possess proper subgroups A_1, B_1 of finite index containing H ; (ii) neither A nor B lies in the variety generated by H ; (iii) $H < A_1 \leq A$ and $H < B_1 \leq B$, where A_1 and B_1 each satisfies a nontrivial identical relation; (iv) H is nilpotent.

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1. Definitions and notation. Throughout the paper our notation will be standard. We use $G = A \star_H B$ to represent the generalized free product of the groups A and B with the amalgamated subgroup H , as in B. H. Neumann [12]. A group G is residually finite if every nontrivial element of G can be excluded from some normal subgroup of finite index in G . An N -group is a finite group in which the normalizer of every nontrivial solvable subgroup is solvable. A group G is called 3-metabelian if every subgroup of G which is generated by 3 elements is metabelian. A variety of groups is the collection of all groups satisfying a given set of identical relations or laws. The core of the subgroup H in G is represented by $K(G, H)$.

An element g of a group G is a near generator of G if there exists a subset S of G such that $|G : \langle S \rangle|$ is infinite, but $|G : \langle g, S \rangle|$ is finite. Thus, an element g of G is a non-near generator of G if for every subset S of G , finiteness of $|G : \langle g, S \rangle|$ implies finiteness of $|G : \langle S \rangle|$. A subgroup M of a group G is nearly maximal in G if $|G : M|$ is infinite, but $|G : N|$ is finite, whenever $M < N \leq G$. That is, M is nearly maximal in G if it is maximal with respect to being of infinite index in G . The set of all non-near generators of a group G forms a characteristic subgroup called the lower near Frattini subgroup of G , denoted by $\lambda(G)$. The intersection of all nearly maximal subgroups of G is called the upper near Frattini subgroup of G , denoted by $\mu(G)$. If there are no nearly maximal subgroups, then $\mu(G) = G$. In general, $\lambda(G) \leq \mu(G)$. If $\lambda(G) = \mu(G)$, then their common value is called the near Frattini subgroup of G , denoted by $\psi(G)$. Definitions concerning the near Frattini subgroup are due to J. B. Riles [13].

2. Background and history. In response to a question raised by N. Itô concerning the existence of maximal subgroups in free products of groups, G. Higman and B. H. Neumann proved that the Frattini subgroup of a free product of (nontrivial) groups is

the trivial group [11, Theorem 2, page 87]. That is, they showed that free products of groups do have maximal subgroups. They extended Itô's question and asked whether a generalized free product of groups necessarily has maximal subgroups. They asked whether or not the Frattini subgroup of a generalized free product of groups is contained in the amalgamated subgroup. These questions have been answered for some certain classes of generalized free products of groups (see [2, 3, 4, 5, 6, 7, 8, 9]).

Similar results for the (lower) near Frattini subgroups of such generalized free products of groups are produced in [2, 3, 4, 5, 6, 7, 8, 9]. In this paper which is motivated by R. B. J. T. Allenby and C. Y. Tang [1], we continue our investigation to produce more results concerning the relationship between the (lower) near Frattini subgroup and the amalgamated subgroup of these generalized free products. In particular, in Section 3 we show that if $G = A \star_H B$ is finitely generated and residually finite, then $\psi(G) \leq H$, provided H satisfies a nontrivial identical relation. Also, when $G = A \star_H B$ is residually finite, we prove that $\lambda(G) \leq H$, if any of the following conditions is satisfied: (i) H satisfies a nontrivial identical relation and A, B possess proper subgroups A_1, B_1 of finite index containing H ; (ii) neither A nor B lies in the variety generated by H ; (iii) $H < A_1 \leq A$ and $H < B_1 \leq B$, where A_1 and B_1 each satisfies a nontrivial identical relation; (iv) H is nilpotent.

3. Results. Before tackling the new results, we need to state some known results from previous works.

THEOREM 3.1 [5, Theorem 3.6, page 502]. *Let $G = A \star_H B$. If H satisfies the minimum condition on subgroups, then $\lambda(G) \leq K(G, H)$.*

THEOREM 3.2 [7, Theorem 4.2, page 6]. *Let $G = A \star_H B$. If there exists a nontrivial normal subgroup N of G such that $N \cap H = 1$, then $\lambda(G) \leq H$.*

PROPOSITION 3.3 [7, Proposition 4.6, page 6]. *Let $G = A \star_H B$. If $\lambda(G) \cap A = \lambda(G) \cap B = \lambda(G) \cap H$, then $\lambda(G) \leq H$.*

THEOREM 3.4 [7, Theorem 4.7, page 6]. *Let $G = A \star_H B$. Suppose A_1 and B_1 are finite normal subgroups of A and B , respectively. If $A_1 \cap H = B_1 \cap H$, and at least one of A_1 or B_1 is not contained in H , then $\lambda(G) \leq H$.*

THEOREM 3.5 [9, Theorem 3.12, page 608]. *Let $G = A \star_H B$ be residually finite. If $|A : H| = |B : H| = 2$, then $\lambda(G) \leq K(G, H)$.*

THEOREM 3.6 [7, Theorem 4.11, page 7]. *Let $G = A \star_H B$. If A and B are countable groups, then $\lambda(G) \leq H$.*

THEOREM 3.7. *Let $G = A \star_H B$ be finitely generated and residually finite. If H satisfies a nontrivial identical relation, then $\psi(G) \leq H$.*

REMARK 3.8. We could refer to Theorem 3.6 and accept Theorem 3.7 without a proof. However, we present a direct proof, independent of the proof of Theorem 3.6.

PROOF. Since G is finitely generated by J. B. Riles [13, Proposition 1, page 157], $\lambda(G) = \mu(G) = \psi(G)$. Therefore, it is enough to show that $\lambda(G) \leq H$. If $\lambda(G) \cap A =$

$\lambda(G) \cap B = \lambda(G) \cap H$, then [Proposition 3.3](#) is applicable. Otherwise, at least one of $\lambda(G) \cap A$ or $\lambda(G) \cap B$ properly contains $\lambda(G) \cap H$. If $|A : H| = |B : H| = 2$, then by [Theorem 3.5](#), $\lambda(G) \leq H$. Therefore, without loss of generality, we may assume that $|A : H| > 2$. Thus, there must exist an element $a \in A$ such that $a \in \lambda(G)$ but $a \notin H$. Also, we let $b \in B \setminus H$ and $a_1 \in A \setminus H \cup aH$. Now, since $a_1^{-1}a \notin H$, we conclude that $u = a_1^{-1}(ab^{-1}ab)a_1 \in \lambda(G)$, where in reduced form the initial and final letters of u are in $A \setminus H$.

Since $\lambda(G)$ is characteristic in G , the rest of the proof is very similar to the proof of Theorem 2 of R. B. J. T. Allenby and C. Y. Tang [1, page 302]. Thus, we use the same notation and set up as in [1] and we replace $\Phi(G)$ by $\lambda(G)$. In particular, we let $S = \langle u, b^{-1}ub \rangle$, $w(x_1, x_2, \dots, x_n)$, $w(y_1, y_2, \dots, y_n)$, N , U , V , $\tilde{A}$, $\tilde{B}$, $\tilde{H}$, $\tilde{G}$, and the natural map ψ , be as in the proof of Theorem 2 of [1]. To complete the proof we use the fact that G is residually finite, [Theorem 3.4](#), as well as the fact that the natural homomorphism takes a non-near generator of G to a non-near generator of $\tilde{G}$. $\square$

[Theorem 3.7](#) can be applied to various residually finite generalized free products of groups. For example, if $G = A \star_H B$ is residually finite and is finitely generated, then $\psi(G) \leq H$, provided: (i) H is of finite exponent, H is periodic or H is an N -group; (ii) H is the ordinary free product of two cyclic groups of order 2; (iii) H is metabelian, or H is 3-metabelian; (iv) H is nilpotent.

THEOREM 3.9. *Let $G = A \star_H B$ be residually finite. If H satisfies a nontrivial identical relation and if A, B possess proper subgroups A_1, B_1 of finite index containing H , then $\lambda(G) \leq H$.*

PROOF. The first part of the proof is similar to the proof of Theorem 3 of R. B. J. T. Allenby and C. Y. Tang [1, page 302], and we note that A_1, B_1 here correspond to K, L in [1]. Thus, if $U = N \cap A \cap \lambda(G) \triangleleft A$ and $V = N \cap B \cap \lambda(G) \triangleleft B$, are as in [1], where $\Phi(G)$ is replaced by $\lambda(G)$, then it is enough to show that $HU \neq A$ and $HV \neq B$. If this is not the case, then without loss of generality, we may assume that $HU = A$. Now, from the fact that $H \leq A_1 < A$ and $|A : A_1| < \infty$, we deduce that $A_1(\lambda(G) \cap A) \geq HU = A$. This implies that

$$A = \langle A_1, \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n \rangle, \quad (3.1)$$

where $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are nontrivial and distinct elements of $\lambda(G)$. Thus,

$$G = \langle A, B \rangle = \langle \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n, A_1, B \rangle. \quad (3.2)$$

Hence,

$$|G : \langle \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n, A_1, B \rangle| < \infty. \quad (3.3)$$

But, since $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are non-near generators of G , we must have $|G : \langle A_1, B \rangle| < \infty$.

However, $|G : \langle A_1, B \rangle| < \infty$, is not possible. For if we take $a \in A \setminus A_1$ and $b \in B \setminus H$, then

$$ab \langle A_1, B \rangle, (ab)^2 \langle A_1, B \rangle, \dots, (ab)^n \langle A_1, B \rangle, \dots \quad (3.4)$$

are incongruent mod $\langle A_1, B \rangle$. That is, $\langle A_1, B \rangle$ has infinitely many distinct cosets in G . Therefore, the assumption that $HU = A$ is reached to a contradiction, and thus, the proof is complete. $\square$

THEOREM 3.10. *Let $G = A \star_H B$ be residually finite. If neither A nor B lies in the variety generated by H , then $\lambda(G) \leq H$.*

PROOF. First we note that H must satisfy a nontrivial identical relation. Otherwise, H generates the variety of all groups, and thus it must contain both A and B , contradicting the statement of the theorem. Also, since G is residually finite, it contains a collection of normal subgroups $N_\lambda (\lambda \in \Lambda)$ of finite index such that $\bigcap_{\lambda \in \Lambda} N_\lambda = 1$. If there exist $\mu, \nu \in \Lambda$ such that $H(A \cap N_\mu) \neq A$ and $H(B \cap N_\nu) \neq B$, then by [Theorem 3.9](#), $\lambda(G) \leq H$. On the other hand, if $H(A \cap N_\lambda) = A$ for all $\lambda \in \Lambda$ and $H(B \cap N_\lambda) = B$, for all $\lambda \in \Lambda$, then again, by the argument given by R. B. J. T. Allenby and C. Y. Tang in the proof of the Frattini version of this theorem [[1](#), Theorem 3, page 303], we conclude that $\lambda(G) \leq H$. If $\lambda(G) \not\leq H$, then we must have either $H(A \cap N_\lambda) = A$ for all $\lambda \in \Lambda$ or $H(B \cap N_\lambda) = B$, for all $\lambda \in \Lambda$, but not both. Hence, either A or B must satisfy the same identical relation as the amalgamated subgroup H , which is impossible, by the statement of the theorem. Therefore, we must have $\lambda(G) \leq H$, as desired. $\square$

THEOREM 3.11. *Let $G = A \star_H B$ be residually finite. If $H < A_1 \leq A$ and $H < B_1 \leq B$, where A_1 and B_1 each satisfies a nontrivial identical relation, then $\lambda(G) \leq H$.*

PROOF. Since H satisfies a nontrivial identical relation, if G is finitely generated, then [Theorem 3.6](#) is applicable. Also, if both A_1 and B_1 are of finite indices in A and B , respectively, then again by [Theorem 3.9](#), $\lambda(G) \leq H$. Now, since both $\lambda(G)$ and $\Phi(G)$ are characteristic subgroups of G , the proof of the general case is very similar to the proof of the Frattini version of this theorem by R. B. J. T. Allenby and C. Y. Tang [[1](#), Theorem 1, page 303], and is left to the reader. $\square$

THEOREM 3.12. *Let $G = A \star_H B$ be residually finite. If H is nilpotent, then $\lambda(G) \leq H$.*

PROOF. If G is finitely generated, then [Theorem 3.7](#) is applicable. Otherwise, we use the same setup and notation as in the proof of the Frattini version of this theorem [[1](#), Theorem 5, page 303] by R. B. J. T. Allenby and C. Y. Tang. To complete the proof, we use the fact that the natural homomorphism ψ takes a non-near generator of G to a non-near generator of its factor group $\bar{G}$, and we apply [Theorem 3.1](#) as well. $\square$

As an immediate consequence of [Theorem 3.12](#) and Theorem 7 of G. Baumslag [[10](#), page 196], we have the following corollary.

COROLLARY 3.13. *Let $G = A \star_H B$. If A and B are free groups and H is cyclic, then $\lambda(G) \leq H$.*

From our study of residually finite generalized free products of groups and their lower near Frattini subgroups in this paper, as well as [[8](#), [9](#)], we suspect that if the amalgamated subgroup satisfies a nontrivial identical relation, then the lower near Frattini subgroup of such generalized free products is contained in the amalgamated subgroup. Therefore, we make the following conjecture.

CONJECTURE 3.14. *Let $G = A \star_H B$ be residually finite. If H satisfies a nontrivial identical relation, then $\lambda(G) \leq H$.*

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