ON SEPARATION AXIOMS IN INTUITIONISTIC TOPOLOGICAL SPACES

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Abstract. The purpose of this paper is to investigate several types of separation axioms in intuitionistic topological spaces, developed by Çoker (2000). After giving some characterizations of $T_1$ and $T_2$ separation axioms in intuitionistic topological spaces, we give interrelations between several types of separation axioms and some counterexamples.

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1. Introduction. After the introduction of the concept of a fuzzy set by Zadeh [15], Atanassov [1, 2] has introduced the concept of intuitionistic fuzzy set. Later Çoker et al. [4, 5, 8] have defined intuitionistic fuzzy topological spaces, intuitionistic sets, and intuitionistic topological spaces in [6, 9, 12].

2. Preliminaries. First we present the fundamental definitions (see Çoker [4]).

Definition 2.1 (see [4]). Let $X$ be a nonempty fixed set. An intuitionistic fuzzy set (IS for short) $A$ is an object having the form $A = \langle X, A_1, A_2 \rangle$, where $A_1$ and $A_2$ are subsets of $X$ satisfying $A_1 \cap A_2 = \emptyset$. The set $A_1$ is called the set of members of $A$, while $A_2$ is called the set of nonmembers of $A$.

Definition 2.2 (see [4]). Let $X$ be a nonempty set and let the IS's $A$ and $B$ be in the form $A = \langle X, A_1, A_2 \rangle$, $B = \langle X, B_1, B_2 \rangle$, respectively. Furthermore, let $\{A_i : i \in J\}$ be an arbitrary family of IS's in $X$, where $A_i = \langle X, A_i^{(1)}, A_i^{(2)} \rangle$. Then

(a) $A \subseteq B$ if and only if $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$;
(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$;
(c) $\bar{A} = \langle X, A_2, A_1 \rangle$;
(d) $\cup A_i = \langle X, \cup A_i^{(1)}, \cap A_i^{(2)} \rangle$;
(e) $\cap A_i = \langle X, \cap A_i^{(1)}, \cup A_i^{(2)} \rangle$;
(f) $\emptyset \cup A_i = \langle X, A_i^{(1)}, A_i^{(2)} \rangle$;
(g) $\emptyset \cap A_i = \langle X, A_i^{(1)}, A_i^{(2)} \rangle$;
(h) $\emptyset \subseteq X, \emptyset, X \rangle$; $X = \langle X, X, \emptyset \rangle$.

Let $X$ be a nonempty set, $p \in X$ a fixed element in $X$, and let $A = \langle X, A_1, A_2 \rangle$ be an IS. The IS $p$ defined by $p = \langle X, \{p\}, \{p\}^c \rangle$ is called an intuitionistic point (IP for short) in $X$. The IS $\emptyset \cup p = \langle X, \emptyset^c \rangle$ is called a vanishing intuitionistic point (VIP for short) in $X$. The IS $p$ is said to be contained in $A(p \subseteq A)$ if and only if $p \in A_1$, and similarly, $p$ is said to be contained in $A(p \subseteq A)$ if and only if $p \notin A_2$. For a
An intuitionistic set (PIS for short) of $A$ in $X$, we may write

$$A = (\cup \{ P : P \in A \}) \cup (\cup \{ \sim \in A \}), \tag{2.1}$$

(cf. [9]) and whenever $A$ is not a proper IS (i.e., if $A$ is not of the form $A = \langle X,A_1,A_2 \rangle$, where $A_1 \cup A_2 \neq X$), then $A = \cup \{ P : P \in A \}$ follows. In general, any IS $A$ in $X$ can be written in the form $A = A \cup A^1$ where $A = \cup \{ P : P \in A \}$ and $A = \cup \{ \sim : \sim \in A \}$. Furthermore it is easy to show that, if $A = \langle X,A_1,A_2 \rangle$, then $A = \langle X,A_1,A^1_2 \rangle$ and $A = \langle X,\emptyset,A_2 \rangle$ (cf. [4, 7]).

**Definition 2.3** (see [4]). Let $X$ and $Y$ be two nonempty sets and $f : X \to Y$ a function, $B = \langle Y,B_1,B_2 \rangle$ an IS in $Y$ and $A = \langle X,A_1,A_2 \rangle$ an IS in $X$. Then the preimage of $B$ under $f$, denoted by $f^{-1}(B)$, is the IS in $X$ defined by $f^{-1}(B) = \langle X,f^{-1}(B_1),f^{-1}(B_2) \rangle$, and the image of $A$ under $f$, denoted by $f(A)$, is the IS in $Y$ defined by $f(A) = \langle Y,f(A_1),f(A_2) \rangle$ where $f(A_2) = (f(A_2^2))^\sim$.

You may find the fundamental properties of preimages and images in [4].

**Definition 2.4** (see [6]). An intuitionistic topology (IT for short) on a nonempty set $X$ is a family $\tau$ of IS's in $X$ containing $\emptyset$, $X$ and closed under finite infima and arbitrary suprema. In this case the pair $(X,\tau)$ is called an intuitionistic topological space (ITS for short) and any IS in $\tau$ is known as an intuitionistic open set (IOS for short) in $X$. The complement $\sim A$ of an IOS $A$ in an ITS $(X,\tau)$ is called an intuitionistic closed set (ICS for short) in $X$.

Let $(X,\tau)$ be an ITS on $X$. Then, we can also construct several other ITS's on $X$ in the following way: $\tau_{0,1} = \{ \{ G : G \in \tau \} \}$ and $\tau_{0,2} = \{ (\langle G : G \in \tau \rangle \}$. Furthermore,

$$\tau_1 = \{ G_1 : G = \langle X,G_1,G_2 \rangle \in \tau \}, \quad \tau_2 = \{ G_2^2 : G = \langle X,G_1,G_2 \rangle \in \tau \} \tag{2.2}$$

are topological spaces in $X$ (cf. [6]).

**Definition 2.5.** Let $A$ and $B$ be two IS's on $X$ and $Y$, respectively. Then the product intuitionistic set (PIS for short) of $A$ and $B$ on $X \times Y$ is defined by $U \times V = \langle (X,Y),A_1 \times B_1, A_2^1 \times B_2^1 \rangle$, where $A = \langle X,A_1,A_2 \rangle$ and $B = \langle Y,B_1,B_2 \rangle$.

If $(X,\tau)$ and $(Y,\Phi)$ are ITS's, then the product topology $\tau \times \Phi$ on $X \times Y$ is the IT generated by the base $\mathcal{B} = \{ A \times B : A \in \tau, B \in \Phi \}$. This is so, because, if $A \times B, C \times D \in \mathcal{B}$, then $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. Let $A = \tau \in \tau, B = \Phi$, and $A = \langle X,A_1,A_2 \rangle$, $B = \langle Y,B_1,B_2 \rangle$. Then we have $\pi_1^{-1}(A) = \langle (x,y),A_1 \times Y,A_2 \times Y \rangle = A \times Y$, $\pi_2^{-1}(B) = \langle (X,Y),X \times B_1,X \times B_2 \rangle = X \times B$, and

$$\pi_1^{-1}(A) \cap \pi_2^{-1}(B) = (A \times Y) \cap (X \times B)$$

\begin{align*}
&= \langle (X,Y),(A_1 \times Y) \cap (X \times B_1),(A_2 \times Y) \cup (X \times B_2) \rangle \\
&= \langle (X,Y),A_1 \times B_1,(A_2 \times Y) \cup (X \times B_2) \rangle \\
&= \langle (X,Y),A_1 \times B_1,(A_2^2 \times B_2^2) \rangle = A \times B. \tag{2.3}
\end{align*}
The definition of “neighborhoods” of IP’s and VIP’s can be found in Coşkun and Çoker [9] and “continuous function” between ITS’s can be found in Çoker [6].

**Lemma 2.6.** The projections \( \pi_1 : X \times Y \to X, \pi_2 : X \times Y \to Y, \pi_1(x,y) = x, \pi_2(x,y) = y \) are continuous.

**Proof.** Let \( A \in \tau \), then \( \pi_1^{-1}(A) = (\langle x,y \rangle, \pi_1^{-1}(A_1), \pi_1^{-1}(A_2)) \). Thus we have \( \pi_1^{-1}(A) = (\langle x,y \rangle, A_1 \times Y, A_2 \times Y) = A \times Y \), that is, \( \pi_1 \) is continuous.

In other words, the product topology \( \tau \times \Phi \) on \( X \times Y \) is indeed the initial topology on \( X \times Y \) with respect to the projections \( \pi_1 : X \times Y \to X \) and \( \pi_2 : X \times Y \to Y \). Here the subbase \( \{\pi_1^{-1}(A), \pi_2^{-1}(B) : A \in \tau, B \in \Phi\} \) generates this product topology and the base \( \mathcal{B} \) is given by

\[
\mathcal{B} = \{ \pi_1^{-1}(A) \cap \pi_2^{-1}(B) : A \in \tau, B \in \Phi \} = \{ A \times B : A \in \tau, B \in \Phi \}. \tag{2.4}
\]

**Definition 2.7.** Given the nonempty set \( X \), we define the diagonal \( \Delta_X \) as the following IS in \( X \times X \):

\[
\Delta_X = \{(x_1,x_2), \{(x_1,x_2) : x_1 = x_2\}, \{(x_1,x_2) : x_1 \neq x_2\}\}. \tag{2.5}
\]

Notice that, if \( X \) and \( Y \) are two nonempty sets and \((p,q) \in X \times Y\) a fixed element in \( X \times Y \), then \((p,q)_-\) is contained in \( U \times V((p,q)_- \in U \times V \) for short) if and only if \((p,q) \in U_1 \times V_1\) and \((p,q)_-\) is contained in \( U \times V((p,q)_- \in U \times V \) for short) if and only if \((p,q) \notin (U_1^c \times V_1^c)^c\), or equivalently \((p,q) \in U_2^c \times V_2^c\).

**Definition 2.8.** Let \( X, Y \) be two nonempty sets and \( f : X \to Y \) a function. The graph of \( f \), denoted by \( \text{GR}(f) \), is defined as the following IS in \( X \times Y \):

\[
\text{GR}(f) = \langle (x,y), \{(x,f(x)) : x \in X\}, \{(x,f(x)) : x \in X\} \rangle. \tag{2.6}
\]

3. Separation axioms in intuitionistic topological spaces. In this section, we present \( T_1 \) and \( T_2 \) separation axioms in ITS’s. The separation axioms \( T_1 \) and \( T_2 \) presented here have certain similarities to those in Bayhan and Çoker [3].

**Definition 3.1.** Let \((X, \tau)\) be an ITS, \((X, \tau)\) is said to be

(a) \( T_1(\text{i}) \) \( \iff \forall x, y \in X \ (x \neq y) \ \exists U, V \in \tau \) such that \( x \in U, y \notin U, \) and \( y \in V, \) \( x \notin V \) (cf. [3, 14]);

(b) \( T_1(\text{ii}) \) \( \iff \forall x, y \in X \ (x \neq y) \ \exists U, V \in \tau \) such that \( x \in U, y \notin U, \) and \( y \in V, \) \( x \notin x \in V \) (cf. [3, 14]);

(c) \( T_1(\text{iii}) \) \( \iff \forall x, y \in X \ (x \neq y) \ \exists U, V \in \tau \) such that \( x \in U \subseteq \tilde{y} \) and \( y \in V \subseteq \tilde{x} \) (cf. [3]);

(d) \( T_1(\text{iv}) \) \( \iff \forall x, y \in X \ (x \neq y) \ \exists U, V \in \tau \) such that \( x \in U \subseteq \tilde{y} \) and \( y \in V \subseteq \tilde{x} \) (cf. [3]);

(e) \( T_1(\text{v}) \) \( \iff \forall x, y \in X \ (x \neq y) \ \exists U, V \in \tau \) such that \( y \notin U \) and \( x \notin V \) (cf. [3]);

(f) \( T_1(\text{vi}) \) \( \iff \forall x, y \in X \ (x \neq y) \ \exists U, V \in \tau \) such that \( y \notin U \) and \( x \notin V \) (cf. [3]);

(g) \( T_1(\text{vii}) \) \( \iff \forall x \in X, \ x \in \tau \)-closed;

(h) \( T_1(\text{viii}) \) \( \iff \forall x \in X, \ x \in \tau \)-closed.
Theorem 3.2. Let \((X, \tau)\) be an ITS, then the following implications are valid:

\[
\begin{align*}
T_1(v) & \iff T_1(vi) \\
T_1(i) & \iff T_1(i) + T_1(ii) \implies T_1(ii) \\
T_1(vii) & \iff T_1(iii) \implies T_1(iv)
\end{align*}
\]

Proof. The proof is obvious. \(\square\)

Counterexample 3.3. Let \(X = \{a, b, c\}\) and define the ITS \(\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}\).

Counterexample 3.4. Let \(X = \{a, b\}\) and define the ITS \(\tau = \{\emptyset, \{a\}, \{b\}\}\) on \(X\), where \(A = \{a\}\), \(B = \{b\}\), \(C = \{a, b\}\), \(D = \{a, c\}\), \(E = \{a, b, c\}\), \(F = \{b, c\}\), \(G = \{a, b, c\}\), \(H = \{a\}\), \(K = \{b\}\). Then \((X, \tau)\) is \(T_1(v)\), but not \(T_1(i)\).

Counterexample 3.5. Let \(X = \{a, b, c\}\) and define the ITS \(\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}\) on \(X\), where \(A = \{a\}\), \(B = \{b\}\), \(C = \{a, b\}\), \(D = \{a, c\}\), \(E = \{a, b, c\}\), \(F = \{b, c\}\). Then \((X, \tau)\) is \(T_1(vi)\), but not \(T_1(i)\).

Counterexample 3.6. Let \(X = \{a, b, c\}\) and define the IS’s \(A = \{\emptyset, \{a\}\}, B = \{\emptyset, \{b\}\}, C = \{\emptyset, \{a\}, \{b\}\}, D = \{\emptyset, \{b\}, \{c\}\}, E = \{\emptyset, \{a\}, \{b\}, \{c\}\}, F = \{\emptyset, \{a\}, \{b\}, \{c\}\}\) on \(X\). Then \((X, \tau)\) is \(T_1(iv)\), but not \(T_1(i)\).

Counterexample 3.7. Let \(X = \{a, b, c\}\) and consider the family \(\tau = \{\emptyset, X, A, B, C, D, E, F, G, H, K\}\), where \(A = \{\emptyset, \{a\}\}, B = \{\emptyset, \{b\}\}, C = \{\emptyset, \{a\}, \{b\}\}, D = \{\emptyset, \{a\}, \{c\}\}, E = \{\emptyset, \{a\}, \{c\}, \{a, b, c\}\}, F = \{\emptyset, \{a\}, \{b, c\}\}, G = \{\emptyset, \{a\}, \{a, b\}\}\) on \(X\). Then the ITS \((X, \tau)\) is \(T_1(v)\), but not \(T_1(i)\).

Counterexample 3.8. Let \(X = \{a, b, c\}\) and consider the family \(\tau = \{\emptyset, X, A, B, C, D, E, F, G, H, K\}\), where \(A = \{\emptyset, \{a\}\}, B = \{\emptyset, \{b\}\}, C = \{\emptyset, \{a\}, \{b, c\}\}, D = \{\emptyset, \{a\}, \{b\}\}, E = \{\emptyset, \{a\}, \{c\}\}, F = \{\emptyset, \{a\}, \{b\}\}, G = \{\emptyset, \{a\}, \{c\}\}\) on \(X\). Then the ITS \((X, \tau)\) on \(X\) is \(T_1(iv)\), but not \(T_1(iii)\).

Counterexample 3.9. Let \(X = \{a, b, c\}\) and consider the family \(\tau = \{\emptyset, X, A, B, C, D, E, F, G\}\), where \(A = \{\emptyset, \{a\}\}, B = \{\emptyset, \{b\}\}, C = \{\emptyset, \{b\}\}, D = \{\emptyset, \{a\}, \{b\}\}, E = \{\emptyset, \{a\}, \{c\}\}, F = \{\emptyset, \{a\}, \{b\}\}, G = \{\emptyset, \{a\}, \{c\}\}\) on \(X\). Then the ITS \((X, \tau)\) on \(X\) is \(T_1(iv)\), but not \(T_1(iii)\).
Then \( \tau = \{ \emptyset, X \} \cup \{ A_n : n = 1, 2, 3, \ldots \} \) is an IT on \( X \). Clearly \((X, \tau)\) is \( T_1(vi) \), but not \( T_1(ii) \).

**Proposition 3.11.** Let \((X, \tau)\) be an ITS. Then
(a) \((X, \tau)\) is \( T_1(i) \) if and only if \((X, \tau_1)\) is \( T_1 \).
(b) \((X, \tau)\) is \( T_1(iii) \) if and only if \((X, \tau_2)\) is \( T_1 \).
(c) \((X, \tau)\) is \( T_1(i) \) if and only if \((X, \tau_{0.1})\) is \( T_1(i) \).
(d) \((X, \tau)\) is \( T_1(ii) \) if and only if \((X, \tau_{0.2})\) is \( T_1(ii) \).

**Definition 3.12.** Let \((X, \tau)\) be an ITS. \((X, \tau)\) is said to be
(a) \( T_2(i) \Leftrightarrow \forall x, y \in X (x \neq y) \exists U, V \in \tau \text{ such that } x \in U, \ y \in V, \text{ and } U \cap V = \emptyset \) (cf. [3, 13]);
(b) \( T_2(ii) \Leftrightarrow \forall x, y \in X (x \neq y) \exists U, V \in \tau \text{ such that } x \in U, \ y \in V, \text{ and } U \cap V = \emptyset \) (cf. [3, 13]);
(c) \( T_2(iii) \Leftrightarrow \forall x, y \in X (x \neq y) \exists U, V \in \tau \text{ such that } x \in U, \ y \in V, \text{ and } U \subseteq V \) (cf. [3, 10]);
(d) \( T_2(iv) \Leftrightarrow \forall x, y \in X (x \neq y) \exists U, V \in \tau \text{ such that } x \in U, \ y \in V, \text{ and } U \subseteq V \) (cf. [3, 10]);
(e) \( T_2(v) \Leftrightarrow \forall x, y \in X (x \neq y) \exists U, V \in \tau \text{ such that } x \in U, \ y \in V, \text{ and } U \subseteq V \) (cf. [3, 11]);
(f) \( T_2(vi) \Leftrightarrow \forall x, y \in X (x \neq y) \exists U, V \in \tau \text{ such that } x \in U, \ y \in V, \text{ and } U \subseteq V \) (cf. [3, 11]);
(g) \( T_2(vii) \Leftrightarrow \Delta_X \) is an ICS in the product ITS \((X \times X, \tau_{X \times X})\).

**Theorem 3.13.** Let \((X, \tau)\) be an ITS. Then the following implications are valid:

\[
\begin{array}{cccc}
T_2(v) & \rightarrow & T_2(vi) & \\
\downarrow & & \downarrow & \\
T_2(vii) & \leftarrow & T_2(i) & \rightarrow \quad T_2(ii) \\
& & \downarrow & \\
& & T_2(iii) & \rightarrow \quad T_2(iv)
\end{array}
\] (3.3)

**Proof.** We prove only the case \( T_2(i) \Rightarrow T_2(vii) \). We must see that \( \Delta_X \) is an IOS in \((X \times X, \tau_{X \times X})\). Let \((x, y)_- \in \Delta_X \). This means that \((x, y) \in \{(x, y) : x \neq y\}\), that is, \( x \neq y \). Since \((X, \tau)\) is \( T_2(i) \), there exist \( U, V \in \tau \) such that \( x \in U, \ y \in V, \text{ and } U \cap V = \emptyset \). Now in this case we have \((x, y)_- \in U \times V \subseteq \Delta_X \). Indeed, from \( x \in U_1 \) and \( y \in V_1 \) we get
\((x,y) \in U_1 \times V_1\), that is, \((x,y)_- \in U \times V\). We also know that \(U \times V \subseteq \Delta_X \iff U_1 \times V_1 \subseteq \{(x,y) : x \neq y\}\) and \((U_1^2 \times V_1^2)^c \subseteq \{(x,y) : x = y\}\). If \((y_1,y_2) \in U_1 \times V_1\), then \(y_1 \in U_1\), \(y_2 \in V_1 \Rightarrow y_1 \neq y_2 \Rightarrow (y_1,y_2) \in \{(x,y) : x \neq y\}\) follows. Thus the first inclusion is true. For the second, \((y_1,y_2) \in U_1^2 \times V_1^2 \Rightarrow y_1 \in U_1^2\) and \(y_2 \in V_1^2 \Rightarrow y_1 \neq y_2\), that is, we have \(U_1^2 \times V_1^2 \subseteq \{(x,y) : x \neq y\}\). Thus we see that \((y_1,y_2) \in \{(x,y) : x = y\}\). The second inclusion is true, too. Now since

\[
\Delta_X = \bigcup_{(y_1,y_2) \in \Delta_X} (y_1,y_2)_- ,
\]

it follows from the fact that \(\Delta_X\) is not a proper IS, that \(\Delta_X\) is an IOS in \((X \times X)\), that is, \((X,\tau)\) is \(T_2(vii)\).

**Counterexample 3.14.** Let \(X = \{a,b\}\) and consider the family \(\tau = \{\emptyset, X, A, B\}\) on \(X\), where \(A = \langle X, \emptyset, \{b\}\rangle\), \(B = \langle X, \emptyset, \{a\}\rangle\). Then the ITS \((X, \tau)\) on \(X\) is \(T_2(ii)\), but not \(T_2(i)\).

**Counterexample 3.15.** Let \(X = \{a,b,c\}\) and define the IS’s \(A = \langle X, \emptyset, \{b,c\}\rangle\), \(B = \langle X, \{b\}, \{a\}\rangle\), \(C = \langle X, \{a\}, \{c\}\rangle\), and \(D = \langle X, \emptyset, \{a,b\}\rangle\). Let \(\tau\) denote the IT on \(X\) generated by the subbase \(S = \{A, B, C, D\}\). Then \((X, \tau)\) is \(T_2(iv)\), but not \(T_2(iii)\).

**Counterexample 3.16.** Let \(X = \{a,b,c,d\}\) and consider the family \(\tau = \{\emptyset, X, A, B, C, D, E, F, G, H, K, L, M\}\) on \(X\), where \(A = \langle X, \emptyset, \{b\}\rangle\), \(B = \langle X, \emptyset, \{a,c\}\rangle\), \(C = \langle X, \{a\}, \{b,c\}\rangle\), \(D = \langle X, \emptyset, \{a\}\rangle\), \(E = \langle X, \emptyset, \{a,b\}\rangle\), \(F = \langle X, \emptyset, \{c\}\rangle\), \(G = \langle X, \{a\}, \{c\}\rangle\), \(H = \langle X, \{a\}, \emptyset\rangle\), \(K = \langle X, \{a\}, \{b\}\rangle\), \(L = \langle X, \emptyset, \{b,c\}\rangle\), and \(M = \langle X, \emptyset, \emptyset\rangle\). Then the ITS \((X, \tau)\) on \(X\) is \(T_2(vi)\), but not \(T_2(v)\).

**Counterexample 3.17.** Let \(X = \{a,b,c,d\}\) and define the IS's \(A = \langle X, \{a\}, \{b\}\rangle\), \(B = \langle X, \{b\}, \{a,d\}\rangle\), \(C = \langle X, \{b\}, \{c\}\rangle\), \(D = \langle X, \{c\}, \{a,b\}\rangle\), \(E = \langle X, \{a\}, \{d\}\rangle\), \(F = \langle X, \{d\}, \{a\}\rangle\), \(G = \langle X, \{b\}, \{d\}\rangle\), \(H = \langle X, \{d\}, \{b\}\rangle\), \(K = \langle X, \{c\}, \{d\}\rangle\), \(L = \langle X, \{d\}, \{c\}\rangle\), \(M = \langle X, \{a\}, \{c\}\rangle\), and \(N = \langle X, \{c\}, \{a\}\rangle\). Let \(\tau\) denote the IT on \(X\) generated by the subbase \(S = \{A, B, C, D, E, F, G, H, K, L, M, N\}\). Then \((X, \tau)\) is \(T_2(iii)\), but not \(T_2(i)\).

**Counterexample 3.18.** Let \(X = \{a,b\}\) and consider the family \(\tau = \{\emptyset, X, A, B\}\) on \(X\), where \(A = \langle X, \emptyset, \{b\}\rangle\), \(B = \langle X, \emptyset, \{a\}\rangle\). Then the ITS \((X, \tau)\) on \(X\) is \(T_2(iv)\), but not \(T_2(ii)\).

**Counterexample 3.19.** We consider the IT on \(X\) as in **Counterexample 3.15**. \((X, \tau)\) is \(T_2(iv)\), but not \(T_2(i)\).

**Counterexample 3.20.** We consider the ITS on \(X\) as in **Counterexample 3.14**. \((X, \tau)\) is \(T_2(ii)\), but not \(T_2(v)\).

**Proposition 3.21.** Let \((X, \tau)\) be an ITS. Then
(a) \((X, \tau)\) is \(T_2(i) \Rightarrow (X, \tau_1)\) is \(T_2\).
(b) \((X, \tau)\) is \(T_2(ii) \Rightarrow (X, \tau_2)\) is \(T_2\).

**Proposition 3.22.** Let \((X, \tau)\) be an ITS. Then
(a) \((X, \tau)\) is \(T_2(i) \Rightarrow (X, \tau_{0,1})\) is \(T_2(i)\).
(b) \((X, \tau)\) is \(T_2(ii) \Rightarrow (X, \tau_{0,2})\) is \(T_2(ii)\).
Theorem 3.23. Let $(X, \tau)$ be an ITS. Then the following implications are valid:

(a) $T_2(i) \Rightarrow T_1(iii)$.
(b) $T_2(ii) \Rightarrow T_1(iii)$.
(c) $T_2(iii) \Rightarrow T_1(iii)$.
(d) $T_2(iv) \Rightarrow T_1(iv)$.
(e) $T_2(v) \Rightarrow T_1(iii)$.
(f) $T_2(vi) \Rightarrow T_1(vi)$.

Proof. The proof is obvious.

Proposition 3.24. Let $(X, \tau)$ be $T_2(i)$. Then every intuitionistic point $x$ is the intersection of all the intuitionistic closed neighborhoods of $x$.

Proof. Let $(X, \tau)$ be $T_2(i)$ and $x \in X$. We denote the intersection of IC neighborhoods of $x$ by the IS $C = (X, C_1, C_2)$. We assume the contrary and suppose that there exists a distinct IP $y$ in $C$, that is, $y \in C_1$.

Case 1. $\{x\} \subseteq C_1$, then there exists $y \in C_1$ such that $x \neq y$. Since $(X, \tau)$ is $T_2(i)$, there exist IOS’s $U, V$ such that $x \in U, y \in V$, and $U \cap V = \emptyset$ which implies that $U \subseteq \bar{V}$. Hence we have $x \in U \subseteq \bar{V}$. Thus $\bar{V}$ is a closed neighborhood of $x$. From our assumption, we get $y \in \bar{V}$. But it is a contradiction, since $V_1 \cap V_2 = \emptyset$. Thus our assumption is false. This means that $C$ consists only of the IP $x$.

Case 2. $\{x\} \subseteq C_2$ and $\{x\} = C_1$, then there exists $y \in C^2$ such that $y \neq x$. Since $(X, \tau)$ is $T_2(i)$, there exist IOS’s $U, V \in \tau$ such that $x \in U, y \in V$, and $V \cap U = \emptyset$ and the same result as in the previous assumption holds in this case, too.

Proposition 3.25. Let $(X, \tau)$ be an ITS, $(Y, \Phi)$ a $T_2(i)$ ITS and $f : (X, \tau) \rightarrow (Y, \Phi)$ a continuous function. Then the graph of $f$ is an ICS in $X \times Y$.

Proof. We must show that $\overline{GR(f)}$ is an IOS in $X \times Y$. Let $(x, y) \in \overline{GR(f)}$. Then $(x, y) \in \{(x, f(x)) : x \in X\}$ which implies that $y \neq f(x)$. Since $(Y, \Phi)$ is $T_2(i)$, there exist $U, V \in \Phi$ such that $y \in U, f(x) \in V$, and $U \cap V = \emptyset$. From the assumption that $f$ is continuous, we see that $f^{-1}(V) = (X, f^{-1}(V_1), f^{-1}(V_2))$ is an open neighborhood of $x$. Also $f^{-1}(V) \times U$ is an open neighborhood of $(x, y)$. It can be shown easily that $f^{-1}(V) \times U \subseteq \overline{GR(f)}$. Since $\overline{GR(f)}$ is not a proper IS in $X \times Y$, our assumption holds, that is, $\overline{GR(f)}$ is an IOS in $X \times Y$.

Proposition 3.26. Let $(X, \tau)$ be an ITS, $(Y, \Phi)$ a $T_2(i)$ ITS and $f : (X, \tau) \rightarrow (Y, \Phi)$ a continuous function. Then the IS $C = \{(x_1, x_2), \{(x_1, x_2) : f(x_1) = f(x_2)\}, \{(x_1, x_2) : f(x_1) \neq f(x_2)\}\}$ in $X \times Y$ is an ICS in $X \times Y$.

Proof. A similar argument as in the proof of Proposition 3.25 can be followed.

Proposition 3.27. Let $(X, \tau)$ and $(Y, \Phi)$ be two ITS’s. Then

(a) If $(X, \tau)$ and $(Y, \Phi)$ are $T_1(i)$, then so is $(X \times Y, \tau \times \Phi)$.
(b) If $(X, \tau)$ and $(Y, \Phi)$ are $T_1(ii)$, then so is $(X \times Y, \tau \times \Phi)$. 
**Proof.** (a) Let \((X, \tau)\) and \((Y, \Phi)\) be \(T_1(i)\). Let \((x_1, y_1), (x_2, y_2) \in X \times Y\) and \((x_1, y_1) \neq (x_2, y_2)\). Now suppose that \(x_1 \neq x_2\). Since \((X, \tau)\) is \(T_1(i)\) then there exist \(U, V \in \tau\) such that \(x_1 \in U, x_2 \notin U\), and \(x_2 \in V, x_1 \notin V\). Then we have IOS’s \(U \times Y = \{(X, Y) : U_1 \subseteq Y, (U_1 \times \emptyset)^c\}\) and \(V \times Y = \{(X, Y), V_1 \times Y, (V_1 \times \emptyset)^c\}\) in \(\tau \times \Phi\) having the properties \((x_1, y_1) \in U \times Y\), \((x_2, y_2) \notin U \times Y\), and \((x_2, y_2) \in V \times Y\), \((x_1, y_1) \notin V \times Y\). We can prove the case \(y_1 = y_2\) similarly. Thus we conclude that \((X \times Y, \tau \times \Phi)\) is \(T_1(i)\).

(b) Similar to the previous one.

**Proposition 3.28.** Let \((X, \tau)\) and \((y, \Phi)\) be two ITS’s. Then

(a) If \((X, \tau)\) and \((Y, \Phi)\) are \(T_2(i)\), then so is \((X \times Y, \tau \times \Phi)\).

(b) If \((X, \tau)\) and \((Y, \Phi)\) are \(T_2(ii)\), then so is \((X \times Y, \tau \times \Phi)\).

(c) If \((X, \tau)\) and \((Y, \Phi)\) are \(T_2(iii)\), then so is \((X \times Y, \tau \times \Phi)\).

(d) If \((X, \tau)\) and \((Y, \Phi)\) are \(T_2(ivii)\), then so is \((X \times Y, \tau \times \Phi)\).

**Proof.** (a) Let \((X, \tau), (Y, \Phi)\) be \(T_2(i)\). Let \((x_1, y_1), (x_2, y_2) \in X \times Y\) and \((x_1, y_1) \neq (x_2, y_2)\) and suppose that \(x_1 \neq x_2\). Since \((X, \tau)\) is \(T_2(i)\) then there exist \(U, V \in \tau\) such that \(x_1 \in U, x_2 \notin U\), and \(U \cap V = \emptyset\). Then we can form the IOS’s \(U \times Y = \{(X, Y), U_1 \times Y, (U_1 \times \emptyset)^c\}\) and \(V \times Y = \{(X, Y), V_1 \times Y, (V_1 \times \emptyset)^c\}\) in \(\tau \times \Phi\) which contains \((x_1, y_1)\) and \((x_2, y_2)\), respectively. Now we must see that \((U \times Y) \cap (V \times Y) = \emptyset\). Indeed,

\[
(U \times Y) \cap (V \times Y) = \{(X, Y), (U_1 \times Y) \cap (V_1 \times Y), (U_1 \times \emptyset)^c \cap (V_1 \times \emptyset)^c\} \\
= \{(X, Y), (U_1 \cap V_1) \times (Y \cap Y), [(U_1 \times Y) \cap (V_1 \times Y)]^c\} \\
= \{(X, Y), \emptyset \times Y, [(U_1 \times (V_1 \times Y)]^c\} \\
= \{(X, Y), \emptyset \times X \times Y = \emptyset.\}
\]

Thus \((X \times Y, \tau \times \Phi)\) is \(T_2(i)\).

(b) Similar to previous one.

(c) Assume that \((X, \tau)\) and \((Y, \Phi)\) are \(T_2(iii)\). Let \((x_1, y_1), (x_2, y_2) \in X \times Y\) and \((x_1, y_1) \neq (x_2, y_2)\). Suppose that \(x_1 \neq x_2\). Since \((X, \tau)\) is \(T_2(iii)\), then there exist \(U, V \in \tau\) such that \(x_1 \in U, x_2 \notin U\), and \(U \cap V = \emptyset\). Then we have IOS’s \(U \times Y = \{(X, Y), U_1 \times Y, (U_1 \times \emptyset)^c\}\) and \(V \times Y = \{(X, Y), V_1 \times Y, (V_1 \times \emptyset)^c\}\) in \(\tau \times \Phi\) containing \((x_1, y_1)\) and \((x_2, y_2)\), respectively. Now, it is easy to see that \(U \times Y \subseteq V \times Y\) holds, which is identical to \(U \times Y \subseteq (V_1 \times Y)^c\) and \(V_1 \times Y \subseteq (U_1 \times Y)^c\). A similar argument holds if \(y_1 \neq y_2\). Thus we conclude that \((X \times Y, \tau \times \Phi)\) is \(T_2(iii)\).

(d) We are to show that \(\Delta_{XY}\) is an ICS, that is, \(\Delta_{XY}\) is an IOS. Since \(\Delta_{XY}\) is not a proper IS in \(X \times Y\), it is sufficient to show that for every \((p_1, q_1), (p_2, q_2)\) \(\in \Delta_{XY}\), there exists an IOS \(S\) in \((X \times Y) \times (X \times Y)\) such that \((p_1, q_1), (p_2, q_2)\) \(\in S \subseteq \Delta_{XY}\). Since \((p_1, q_1), (p_2, q_2)\) \(\in \Delta_{XY}\), we get \((p_1, q_1) \neq (p_2, q_2)\), that is, \(p_1 \neq p_2\) or \(q_1 \neq q_2\). Here come three possible cases:

1. \(p_1 \neq p_2, q_1 = q_2\);
2. \(p_1 = p_2, q_1 \neq q_2\);
3. \(p_1 \neq p_2, q_1 \neq q_2\).

Here we show only case (3). Other cases can be proved similarly. Let \(p_1 \neq p_2, q_1 \neq q_2\). Since \((p_1, p_2) \in \Delta_X, (q_1, q_2) \in \Delta_Y\) and \(\Delta_X, \Delta_Y\) are IOS’s, \(\exists U_1, U_2 \in \tau\) and \(V_1, V_2\),

\[\text{...} \]
V_2 \in \Phi \) such that \((p_1, p_2)_- \in U_1 \times U_2 \subseteq \bar{\Delta}_X \) and \((q_1, q_2)_- \in V_1 \times V_2 \subseteq \bar{\Delta}_Y \). We prove that \(((p_1, q_1), (p_2, q_2))_- \in (U_1 \times V_1) \times (U_2 \times V_2) \subseteq \bar{\Delta}_{X \times Y} \). This can be shown in two steps.

**Step 1.** The expression \(((p_1, q_1), (p_2, q_2))_- \in (U_1 \times V_1) \times (U_2 \times V_2) \) is equivalent to \(((p_1, q_1), (p_2, q_2)) \subseteq ((p_1, q_1), (p_2, q_2)) \subseteq (U_1 \times V_1) \times (U_2 \times V_2) \). This means that \((p_1, q_1) \in U_1 \times V_1 \) and \((p_2, q_2) \in U_2 \times V_2 \) which are true, since \(p_1 \in U_1 \), \(p_2 \in U_2 \), \(q_1 \in V_1 \), \(q_2 \in V_2 \).

**Step 2.** We show the inclusion \((U_1 \times V_1) \times (U_2 \times V_2) \subseteq \bar{\Delta}_{X \times Y} \). For this purpose we must first show that \((U_1 \times V_1) \subseteq ((u_1, v_1), (u_2, v_2)) \) or equivalently, \((U_1 \times V_1) \subseteq ((u_1, v_1), (u_2, v_2)) \). This is true since \(U_1 \times U_2 \subseteq \bar{\Delta}_X \) and \(V_1 \times V_2 \subseteq \bar{\Delta}_Y \), we have \(U_1 \times U_2 \subseteq ((u_1, u_2), (u_2, v_2)) \) or \(V_1 \times V_2 \subseteq ((u_1, v_1), (u_1, v_2)) \). Thus the first inclusion is true. The second inclusion can be proved similarly. Hence \(\bar{\Delta}_{X \times Y} \) is an ICS, which means that \((X, Y, \tau \times \Phi) \) is \(T_2(iv) \).

**Remark 3.29.** Let \((X, \tau) \) and \((Y, \Phi) \) be \(T_2(iii) \). Then \((\times \times \times \times, \tau \times \Phi) \) may not be \(T_2(iv) \).

Here come the reverse implications.

**Proposition 3.30.** Let \((X, \tau) \) and \((Y, \Phi) \) be two ITS's. Then
(a) If \((\times \times \times \times, \tau \times \Phi) \) is \(T_2(i) \), then so are \((X, \tau) \) and \((Y, \Phi) \).
(b) If \((X, \tau, \times \Phi) \) is \(T_2(iii) \), then so are \((X, \tau) \) and \((Y, \Phi) \).
(c) If \((X, \times \times \times \times, \tau \times \Phi) \) is \(T_2(iii) \), then so are \((X, \tau) \) and \((Y, \Phi) \).

**Proof.** The proofs of (a) and (b) are easy. (c) Let \((\times \times \times \times, \tau \times \Phi) \) be \(T_2(iii) \), and \(x_1 \neq x_2 \) \((x_1, x_2) \in X \). We take a fixed \(y \in Y \). Then, since \((x_1, y) \neq (x_2, y)\) and \(X \times Y \) is \(T_2(iii) \), there exist \(U_1 = Z_1 \subseteq \bar{\Delta}_X \) and \(V \subseteq \bar{\Delta}_X \) such that \((x_1, y) \in U_1 \times V \) and \((x_2, y) \in \bar{T}_1 \times U_1 \). From the last intersection we get \((U_1 \times V) \subseteq (Z_1 \times \bar{T}_1) \) and \((U_1 \times V) \subseteq (Z_1 \times \bar{T}_1) \). Thus \(U_1 \subseteq U_2 \). Similarly \(y \in T_1 \) and \(V \subseteq \bar{T}_1 \) implies that \(Z_1 \subseteq \bar{T}_1 \) and \(U_1 \subseteq \bar{T}_1 \). Similarly \((Y, \Phi) \) is \(T_2(iii) \), too.

**References**


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