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a-MINIMAL SETS AND RELATED TOPICS IN TRANSFORMATION SEMIGROUPS (II)

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ABSTRACT. We give some generalizations of proximal relation and distal structure relation of a transformation semigroup in terms of A-$\overline{\text{minimal}}$ sets and A-$\overline{\overline{\text{minimal}}}$ sets instead of minimal right ideals and conclude similar results.

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1. Preliminaries. By a transformation semigroup (X,S,ρ) (or simply (X,S)) we mean a compact Hausdorff topological space X, a discrete topological semigroup S with identity e, and a continuous map $\rho : X\times S \to X$ ($\rho(x,s) = xs\,\forall x \in X, \forall s \in S$), such that

(1) $xe = x \;\forall x \in X$;
(2) $x(st) = (xs)t \;\forall x \in X,\ \forall s,t \in S$.

In the transformation semigroup (X,S), for each $s\in S$ define $\pi^s : X \to X$ by $\pi^s(x) = xs$ ($\forall x \in X$). We assume the semigroup S acts effectively on X, that is, for each $s,t\in S$, $s \neq t$ if and only if $\pi^s \neq \pi^t$. The closure of $\{\pi^s \mid s \in S\}$ in X^X (with pointwise convergence topology) is called the enveloping semigroup (or Ellis semigroup) of (X,S) and is denoted by $\mathrm{E}(X,S)$ (or simply $\mathrm{E}(X)$), $\mathrm{E}(X)$ has a semigroup structure [1]. A nonempty subset I of $\mathrm{E}(X)$ is called a right ideal of $\mathrm{E}(X)$ if $I\mathrm{E}(X) \subseteq I$, moreover, if the right ideal I of $\mathrm{E}(X)$ does not have any proper subset which is a right ideal of $\mathrm{E}(X)$, then I is called a minimal right ideal of $\mathrm{E}(X)$, the set of all minimal right ideals of $\mathrm{E}(X)$ is denoted by $\mathrm{Min}(\mathrm{E}(X))$. An element u of $\mathrm{E}(X)$ is called idempotent if $u^2 = u$. For $p \in \mathrm{E}(X)$ and $a \in X$ the maps $\mathrm{L}_p : \mathrm{E}(X) \to \mathrm{E}(X)$ and $\theta_a : \mathrm{E}(X) \to X$ defined by $\mathrm{L}_p(q) = pq$ and $\theta_a(q) = aq$ ($q \in \mathrm{E}(X)$), respectively, are continuous [2, Propositions 3.2 and 3.3]. Let I be a right ideal of $\mathrm{E}(X)$, $B \subseteq \mathrm{E}(X)$, $C \subseteq X$ ($B,C \neq \emptyset$) and $a \in X$. Standing notations:

$$\begin{aligned}
&\mathrm{S}(I) = \{p \in I \mid \mathrm{L}_p : I \to I \text{ is surjective}\}, && F(a,B) = \{p \in B \mid ap = a\},\\
&\mathrm{I}(I) = \{p \in I \mid \mathrm{L}_p : I \to I \text{ is injective}\}, && F(C,B) = \bigcap_{c\in C} F(c,B),\\
&\mathrm{B}(I) = \{p \in I \mid \mathrm{L}_p : I \to I \text{ is bijective}\}, && \overline{F}(C,B) = \{p \in B \mid Cp = C\},\\
&J(B) = \{u \in B \mid u^2 = u\}.
\end{aligned} \tag{1.1}$$

A nonempty subset Z of X is called invariant if $ZS \subseteq Z$, moreover, a closed invariant subset Z of X is called minimal if it does not have any proper closed invariant subset. Also $a \in X$ is called almost periodic if $\overline{aS} = a\mathrm{E}(X)$ is a minimal subset of X [3, Theorems 1.15 and 1.17]. Let K be a closed right ideal of $\mathrm{E}(X)$, then K is called an A-$\overline{\text{minimal}}$ set if for each $b \in A$, $bK = b\mathrm{E}(X)$ and K does not contain any closed right ideal L of $\mathrm{E}(X)$ such that $K \neq L$ and for each $b \in A$, $bL = b\mathrm{E}(X)$, also K is called an A-$\overline{\overline{\text{minimal}}}$ set if $AK = A\mathrm{E}(X)$ and K does not contain any closed right ideal L of $\mathrm{E}(X)$ such that $K \neq L$ and $AL = A\mathrm{E}(X)$; the collection of all A-$\overline{\text{minimal}}$ sets is denoted by $\overline{\mathrm{M}}_{(X,S)}(A)$ or simply $\overline{\mathrm{M}}(A)$ and the collection of all A-$\overline{\overline{\text{minimal}}}$ sets is denoted by $\overline{\overline{\mathrm{M}}}_{(X,S)}(A)$ or simply $\overline{\overline{\mathrm{M}}}(A)$; we use $\mathrm{M}_{(X,S)}(a)$ (or simply $\mathrm{M}(a)$) instead of $\overline{\mathrm{M}}_{(X,S)}(\{a\})$ and its elements are called a-minimal sets; in addition we introduce the following sets:

$$
\begin{aligned}
\overline{\mathcal{M}}(X,S) &= \{D \subseteq X \mid D \neq \varnothing,\ \forall K \in \overline{\mathrm{M}}(D)\ J(F(D,K)) \neq \varnothing\},\\
\overline{\overline{\mathcal{M}}}(X,S) &= \{D \subseteq X \mid D \neq \varnothing,\ \overline{\overline{\mathrm{M}}}(D) \neq \varnothing,\ \forall K \in \overline{\overline{\mathrm{M}}}(D)\ J(F(D,K)) \neq \varnothing\},
\end{aligned}
\tag{1.2}
$$

the transformation semigroup (X,S) is called $A\overset{(-)}{-}$distal (or simply A-distal) if for each $b \in A$, $\mathrm{E}(X) \in \mathrm{M}(b)$, and it is called $A\overset{(\overline{\mathrm{M}})}{-}$distal (respectively, $A\overset{(\overline{\overline{\mathrm{M}}})}{-}$distal) if $\mathrm{E}(X) \in \overline{\mathrm{M}}(A)$ (respectively, $\mathrm{E}(X) \in \overline{\overline{\mathrm{M}}}(A)$).

Let (X,S) and (Y,S) be transformation semigroups, then the continuous map $\varphi : (X,S) \to (Y,S)$ is called a homomorphism if $\varphi(xs) = \varphi(x)s$ ($\forall x \in X,\ \forall s \in S$), if φ is onto, then there exists a unique induced homomorphism $\hat{\varphi} : (\mathrm{E}(X),S) \to (\mathrm{E}(Y),S)$ which is onto and for each $x \in X$, the following diagram commutes:

$$
\begin{tikzcd}
(\mathrm{E}(X),S) \arrow[r, "\hat{\varphi}"] \arrow[d, "\theta_x"'] & (E(Y),S) \arrow[d, "\theta_{\varphi(x)}"] \\
(X,S) \arrow[r, "\varphi"] & (Y,S)
\end{tikzcd}
\tag{1.3}
$$

moreover, $\hat{\varphi}$ is a semigroup homomorphism; if φ is onto and one-to-one, it is called an isomorphism, and $\hat{\varphi}$ is an isomorphism too [2, Proposition 3.8]. An equivalence relation $\mathfrak{R}$ on X is called invariant if $\mathfrak{R}$ is an invariant subset of the transformation semigroup $(X \times X, S)$. Let $\mathfrak{R}$ be an equivalence relation on X, then $\pi_{\mathfrak{R}} : X \to X/\mathfrak{R}$ ($\pi_{\mathfrak{R}}(x) = [x]_{\mathfrak{R}}$ ($\forall x \in X$)) is the natural canonical map.

For the remainder of this paper (X,S) is a fixed transformation semigroup, with e as the identity element of S and $\Delta_A = \{(x,x) \mid x \in A\}$.

Definition 1.1. Let A be a nonempty subset of X and let

$$
\begin{aligned}
\mathfrak{I} = \{\mathfrak{R} \mid\ & \mathfrak{R} \text{ is a closed invariant equivalence relation on } X \text{ such that}\\
& (X/\mathfrak{R},S) \text{ is distal}\},\\
\mathfrak{I}_0 = \{\mathfrak{R} \mid\ & \mathfrak{R} \text{ is a closed invariant equivalence relation on } X \text{ such that } (X/\mathfrak{R},S)\\
& \text{is } [A]_{\mathfrak{R}}\text{-distal}\},\\
\mathfrak{I}_1 = \{\mathfrak{R} \mid\ & \mathfrak{R} \text{ is a closed invariant equivalence relation on } X \text{ such that } (X/\mathfrak{R},S)\\
& \text{is } [A]_{\mathfrak{R}}\overset{(\overline{\mathrm{M}})}{-}\text{distal}\},\\
\mathfrak{I}_2 = \{\mathfrak{R} \mid\ & \mathfrak{R} \text{ is a closed invariant equivalence relation on } X \text{ such that } (X/\mathfrak{R},S)\\
& \text{is } [A]_{\mathfrak{R}}\overset{(\overline{\overline{\mathrm{M}}})}{-}\text{distal}\},
\end{aligned}
\tag{1.4}
$$

then $\bigcap_{\mathfrak{R}\in\mathfrak{I}}\mathfrak{R}$, $\bigcap_{\mathfrak{R}\in\mathfrak{I}_0}\mathfrak{R}$, $\bigcap_{\mathfrak{R}\in\mathfrak{I}_1}\mathfrak{R}$, and $\bigcap_{\mathfrak{R}\in\mathfrak{I}_2}\mathfrak{R}$ are called, respectively, proximal structure relation, $A\underline{\overset{(-)}{\quad}}$proximal structure relation (or simply A-proximal structure relation), $A\underline{\overset{(\overline{\mathrm{M}})}{\quad}}$proximal structure relation, and $A\underline{\overset{(\overline{\overline{\mathrm{M}}})}{\quad}}$proximal structure relation (on X), for $a\in X$, instead of "$\{a\}$-proximal structure relation" we simply use "a-proximal structure relation"; and the sets

$$\begin{aligned}
\mathrm{P}(X,S) &= \{(x,y)\in X\times X \mid \exists I\in \mathrm{Min}(\mathrm{E}(X))\ \forall p\in I\ xp=yp\}\\
&\qquad\qquad\qquad\qquad \text{(or simply } \mathrm{P}(X) \text{ or } P\text{)},\\
\mathrm{P}_A(X,S) &= \{(x,y)\in X\times X \mid \exists b\in A\ \exists I\in \mathrm{M}(b)\ \forall p\in I\ xp=yp\}\\
&\qquad\qquad\qquad\qquad \text{(or simply } \mathrm{P}_A(X) \text{ or } \mathrm{P}_A\text{)},\\
\overline{\mathrm{P}}_A(X,S) &= \{(x,y)\in X\times X \mid \exists I\in \overline{\mathrm{M}}(A)\ \forall p\in I\ xp=yp\}\\
&\qquad\qquad\qquad\qquad \text{(or simply } \overline{\mathrm{P}}_A(X) \text{ or } \overline{\mathrm{P}}_A\text{)},\\
\overline{\overline{\mathrm{P}}}_A(X,S) &= \{(x,y)\in X\times X \mid \exists I\in \overline{\overline{\mathrm{M}}}(A)\ \forall p\in I\ xp=yp\}\\
&\qquad\qquad\qquad\qquad \text{(or simply } \overline{\overline{\mathrm{P}}}_A(X) \text{ or } \overline{\overline{\mathrm{P}}}_A\text{)},
\end{aligned} \tag{1.5}$$

are called, respectively, proximal relation, $A\underline{\overset{(-)}{\quad}}$proximal relation (or simply A-proximal relation), $A\underline{\overset{(\overline{\mathrm{M}})}{\quad}}$proximal relation, and $A\underline{\overset{(\overline{\overline{\mathrm{M}}})}{\quad}}$proximal relation (on X), if $a\in X$, then instead of "$\{a\}$-proximal relation" (respectively, "$\mathrm{P}_{\{a\}}(X)$") we simply use "a-proximal relation" (respectively, "$\mathrm{P}_a(X)$").

Theorem 1.2. *Let A be a nonempty subset of X, then by Definition 1.1, we have*

(a) (i) *if $\{\mathfrak{R}_\alpha\}_{\alpha\in\Gamma}$ is a nonempty collection in $\mathfrak{I}$, then $\bigcap_{\alpha\in\Gamma}\mathfrak{R}_\alpha\in\mathfrak{I}$,*

(ii) *if $\{\mathfrak{R}_\alpha\}_{\alpha\in\Gamma}$ is a nonempty collection in $\mathfrak{I}_0$, then $\bigcap_{\alpha\in\Gamma}\mathfrak{R}_\alpha\in\mathfrak{I}_0$,*

(iii) *if $\{\mathfrak{R}_\alpha\}_{\alpha\in\Gamma}$ is a nonempty collection in $\mathfrak{I}_1$ such that for $Z=\{([x_\alpha]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid x\in X\}\subseteq\prod_{\alpha\in\Gamma}X/\mathfrak{R}_\alpha$ we have $\{([a]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid a\in A\}\in\overline{\mathcal{M}}(Z,S)$, then $\bigcap_{\alpha\in\Gamma}\mathfrak{R}_\alpha\in\mathfrak{I}_1$,*

(iv) *if $\{\mathfrak{R}_\alpha\}_{\alpha\in\Gamma}$ is a nonempty collection in $\mathfrak{I}_2$ such that for $Z=\{([x_\alpha]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid x\in X\}\subseteq\prod_{\alpha\in\Gamma}X/\mathfrak{R}_\alpha$ we have $\{([a]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid a\in A\}\in\overline{\overline{\mathcal{M}}}(Z,S)$, then $\bigcap_{\alpha\in\Gamma}\mathfrak{R}_\alpha\in\mathfrak{I}_2$,*

(b) (i) *$X\times X\in\mathfrak{I}\cap\mathfrak{I}_0\cap\mathfrak{I}_1\cap\mathfrak{I}_2$,*

(ii) *$\bigcap_{\mathfrak{R}\in\mathfrak{I}}\mathfrak{R}\in\mathfrak{I}$, $\bigcap_{\mathfrak{R}\in\mathfrak{I}_0}\mathfrak{R}\in\mathfrak{I}_0$,*

(c) (i) *(X,S) is distal if and only if $\Delta_X\in\mathfrak{I}$,*

(ii) *(X,S) is A-distal if and only if $\Delta_X\in\mathfrak{I}_0$,*

(iii) *(X,S) is $A\underline{\overset{(\overline{\mathrm{M}})}{\quad}}$distal if and only if $\Delta_X\in\mathfrak{I}_1$,*

(iv) *(X,S) is $A\underline{\overset{(\overline{\overline{\mathrm{M}}})}{\quad}}$distal if and only if $\Delta_X\in\mathfrak{I}_2$.*

Proof. (a) (ii) Let $\{\mathfrak{R}_\alpha\}_{\alpha\in\Gamma}$ be a nonempty collection in $\mathfrak{I}_0$, then for each $\alpha\in\Gamma$, $(X/\mathfrak{R}_\alpha,S)$ is $[A]_{\mathfrak{R}_\alpha}$-distal, thus $(\prod_{\alpha\in\Gamma}X/\mathfrak{R}_\alpha,S)$ is $\{([a]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid a\in A\}$-distal (since $\{([a]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid a\in A\}\subseteq\prod_{\alpha\in\Gamma}[A]_{\mathfrak{R}_\alpha}$), but $\{([a]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid a\in A\}\subseteq\{([x]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid x\in X\}$ and $\{([x]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid x\in X\}$ is a closed invariant subset of $\prod_{\alpha\in\Gamma}X/\mathfrak{R}_\alpha$, therefore $(\{([x]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid x\in X\},S)$ is $\{([a]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid a\in A\}$-distal [4, Theorem 1.23(c)]. On the other hand, $\varphi:(X/\cap_{\alpha\in\Gamma}\mathfrak{R}_\alpha,S)\to(\{([x]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid x\in X\},S)$ defined by $\varphi([x]_{\cap_{\alpha\in\Gamma}\mathfrak{R}_\alpha})=([x]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}$ $(\forall x\in X)$ is an isomorphism and $(\{([x]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid x\in X\},S)$ is

$\varphi([A]_{\bigcap_{\alpha\in\Gamma}\mathfrak{R}_\alpha})$-distal, therefore $(X/\bigcap_{\alpha\in\Gamma}\mathfrak{R}_\alpha,S)$ is $[A]_{\bigcap_{\alpha\in\Gamma}\mathfrak{R}_\alpha}$-distal and $\bigcap_{\alpha\in\Gamma}\mathfrak{R}_\alpha \in \mathfrak{I}_0$.

(iii) Let $\{\mathfrak{R}_\alpha\}_{\alpha\in\Gamma}$ be a nonempty collection in $\mathfrak{I}_1$, then for each $\alpha\in\Gamma$, $(X/\mathfrak{R}_\alpha,S)$ is $[A]_{\mathfrak{R}_\alpha}\overset{(\overline{\mathrm{M}})}{-}$distal, thus for each $\alpha\in\Gamma$, $J(F([A]_{\mathfrak{R}_\alpha},\mathrm{E}(X/\mathfrak{R}_\alpha)))=\{e\}$, therefore $J(F(\{([a]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid a\in A\},\mathrm{E}(\prod_{\alpha\in\Gamma}X/\mathfrak{R}_\alpha)))=\{e\}$, but $\{([x]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid x\in X\}$ is a closed invariant subset of $\prod_{\alpha\in\Gamma}X/\mathfrak{R}_\alpha$, and by the hypothesis $\{([a]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid a\in A\}\in\overline{\mathcal{M}}(\{([x]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid x\in X\},S)$, thus $(\{([x]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid x\in X\},S)$ is $\{([a]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid a\in A\}\overset{(\overline{\mathrm{M}})}{-}$distal [4, Theorem 1.23(d)]. On the other hand, $\varphi:(X/\bigcap_{\alpha\in\Gamma}\mathfrak{R}_\alpha,S)\to(\{([x]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid x\in X\},S)$ defined by $\varphi([x]_{\bigcap_{\alpha\in\Gamma}\mathfrak{R}_\alpha})=([x]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}$ $(\forall x\in X)$ is an isomorphism, and $(\{([x]_{\mathfrak{R}_\alpha})_{\alpha\in\Gamma}\mid x\in X\},S)$ is $\varphi([A]_{cap_{\alpha\in\Gamma}\mathfrak{R}_\alpha})\overset{(\overline{\mathrm{M}})}{-}$distal, therefore $(X/\bigcap_{\alpha\in\Gamma}\mathfrak{R}_\alpha,S)$ is $[A]_{\bigcap_{\alpha\in\Gamma}\mathfrak{R}_\alpha}\overset{(\overline{\mathrm{M}})}{-}$distal and $\bigcap_{\alpha\in\Gamma}\mathfrak{R}_\alpha\in\mathfrak{I}_1$.

(iv) The proof is similar to (iii).

(b) Let $\mathfrak{R}=X\times X$, then $X/\mathfrak{R}$ is singleton, thus it is clear that $(X/\mathfrak{R},S)$ is distal, $[A]_\mathfrak{R}$-distal, $[A]_\mathfrak{R}\overset{(\overline{\mathrm{M}})}{-}$distal, and $[A]_\mathfrak{R}\overset{(\overline{\overline{\mathrm{M}}})}{-}$distal, thus $X\times X=\mathfrak{R}\in\mathfrak{I}\cap\mathfrak{I}_0\cap\mathfrak{I}_1\cap\mathfrak{I}_2$. On the other hand, by (a) ((i) and (ii)) we have $\bigcap_{\mathfrak{R}\in\mathfrak{I}}\mathfrak{R}\in\mathfrak{I}$ and $\bigcap_{\mathfrak{R}\in\mathfrak{I}_0}\mathfrak{R}\in\mathfrak{I}_0$.

(c) (ii) Let $\mathfrak{R}=\Delta_X$, then the canonical map $\pi_\mathfrak{R}:(X,S)\to(X/\mathfrak{R},S)$ is an isomorphism, thus (X,S) is A-distal if and only if $(X/\mathfrak{R},S)$ is $[A]_\mathfrak{R}$-distal if and only if $\Delta_X=\mathfrak{R}\in\mathfrak{I}_0$. □

Note 1.3. Let A be a nonempty subset of X, then

(a) (i) $\mathrm{P}(X)$ is a reflexive and symmetric relation on X,
   (ii) $\mathrm{P}_A(X)$ is a reflexive and symmetric relation on X,
   (iii) $\overline{\mathrm{P}}_A(X)$ is a reflexive and symmetric relation on X,
   (iv) if $\overline{\overline{\mathrm{M}}}(A)\neq\varnothing$, then $\overline{\overline{\mathrm{P}}}_A(X)$ is a reflexive and symmetric relation on X,

(b) if S is abelian, then $\mathrm{P}(X)$, $\mathrm{P}_A(X)$, $\overline{\mathrm{P}}_A(X)$, and $\overline{\overline{\mathrm{P}}}_A(X)$ (this latter case when $\overline{\overline{\mathrm{M}}}(A)\neq\varnothing$) are invariant relations on X,

(c) (i) for each nonempty subset B of A we have $\overline{\mathrm{P}}_A(X)\subseteq\overline{\mathrm{P}}_B(X)\subseteq\mathrm{P}_B(X)\subseteq\mathrm{P}_A(X)\subseteq\mathrm{P}(X)$,
   (ii) $\mathrm{P}_A(X)=\cup_{a\in A}\mathrm{P}_a(X)$,
   (iii) $\overline{\overline{\mathrm{P}}}_A(X)\subseteq\mathrm{P}(X)$,

(d) (i) if all of the points of A are almost periodic, then $\overline{\overline{\mathrm{P}}}_A(X)=\overline{\mathrm{P}}_A(X)=\mathrm{P}_A(X)=\mathrm{P}(X)$,
   (ii) $\mathrm{P}_A(X)=\Delta_X\Leftrightarrow(\forall a\in A\ \mathrm{P}_a(X)=\Delta_X)$,
   (iii) $\mathrm{P}_A(X)=\Delta_X\Rightarrow(\forall a\in A\ \forall K\in\mathrm{M}(a)\ J(F(a,K))=J(\mathrm{S}(K))=\{e\})$,
   (iv) $\overline{\mathrm{P}}_A(X)=\Delta_X\Rightarrow(\forall K\in\overline{\mathrm{M}}(A)\ J(F(A,K))\subseteq J(\mathrm{S}(K))\subseteq\{e\})$,
   (v) $\overline{\overline{\mathrm{P}}}_A(X)=\Delta_X\Rightarrow(\forall K\in\overline{\overline{\mathrm{M}}}(A)\ J(F(A,K))\subseteq J(\mathrm{S}(K))\subseteq\{e\})$.

Proof. (a) and (b) are clear.

(c) (i) Let B be a nonempty subset of A, then for each $(x,y)\in X\times X$ we have

$$\begin{aligned}
&(x,y)\in\overline{\mathrm{P}}_A(X)\\
&\Longrightarrow \exists K\in\overline{\mathrm{M}}(A)\ \forall p\in K,\ xp=yp\\
&\Longrightarrow \exists K\in\overline{\mathrm{M}}(A)\ \exists L\in\overline{\mathrm{M}}(B)\ \forall p\in K,\ (xp=yp\wedge L\subseteq K)\ \text{(by [4, Corollary 1.3])}\\
&\Longrightarrow \exists L\in\overline{\mathrm{M}}(B)\ \forall p\in L,\ xp=yp\\
&\Longrightarrow (x,y)\in\overline{\mathrm{P}}_B(X)
\end{aligned}$$

$$
\begin{aligned}
&(x,y)\in\overline{\mathrm{P}}_B(X)\\
&\quad\Longrightarrow \exists L\in\overline{\mathrm{M}}(B)\ \forall p\in L,\ xp=yp\\
&\quad\Longrightarrow \exists L\in\overline{\mathrm{M}}(B)\ \forall b\in B\ \exists K\in\mathrm{M}(b)\ \forall p\in L\\
&\qquad\quad (xp=yp\wedge K\subseteq L)\ \text{(by [4, Corollary 1.3])}\\
&\quad\Longrightarrow \forall b\in B\ \exists K\in\mathrm{M}(b)\ \forall p\in K,\ xp=yp\\
&\quad\Longrightarrow \exists b\in B\ \exists K\in\mathrm{M}(b)\ \forall p\in K,\ xp=yp\\
&\quad\Longrightarrow (x,y)\in\mathrm{P}_B(X)\\
&(x,y)\in\mathrm{P}_B(X)\\
&\quad\Longrightarrow \exists b\in B\ \exists K\in\mathrm{M}(b)\ \forall p\in K,\ xp=yp\\
&\quad\Longrightarrow \exists a\in A\ \exists K\in\mathrm{M}(a)\ \forall p\in K,\ xp=yp\\
&\quad\Longrightarrow (x,y)\in\mathrm{P}_A(X)\\
&(x,y)\in\mathrm{P}_A(X)\\
&\quad\Longrightarrow \exists a\in A\ \exists K\in\mathrm{M}(a)\ \forall p\in K,\ xp=yp\\
&\quad\Longrightarrow \exists a\in A\ \exists K\in\mathrm{M}(a)\ \exists L\in\mathrm{Min}(\mathrm{E}(X))\ \forall p\in K,\ (xp=yp\wedge L\subseteq K)\\
&\quad\Longrightarrow \exists L\in\mathrm{Min}(\mathrm{E}(X))\ \forall p\in L,\ xp=yp\\
&\quad\Longrightarrow (x,y)\in\mathrm{P}(X).
\end{aligned}
\tag{1.6}
$$

(ii) It is clear by Definition 1.1.

(iii) It is clear by Definition 1.1 and the fact that each closed right ideal of $\mathrm{E}(X)$ contains at least one element of $\mathrm{Min}(\mathrm{E}(X))$.

(d) (i) If all of the points of A are almost periodic, then for each $a\in A$, $\mathrm{M}(a)=\overline{\mathrm{M}}(A)=\overline{\overline{\mathrm{M}}}(A)=\mathrm{Min}(\mathrm{E}(X))$ [4, Note 1.12], thus $\overline{\overline{\mathrm{P}}}_A(X)=\overline{\mathrm{P}}_A(X)=\mathrm{P}_A(X)=\mathrm{P}(X)$.

(ii) Since for each $a\in A$, $\Delta_X\subseteq\mathrm{P}_a(X)\subseteq\mathrm{P}_A(X)=\cup_{b\in A}\mathrm{P}_b(X)$ (use (c) (ii)), thus $\mathrm{P}_A(X)=\Delta_X$ if and only if for each $a\in A$, $\mathrm{P}_a(X)=\Delta_X$.

(iv) Let $\overline{\mathrm{P}}_A(X)=\Delta_X$ and $K\in\overline{\mathrm{M}}(A)$, then $J(F(A,K))\subseteq J(\mathrm{S}(K))$ [4, Corollary 1.5(Table 1.3)], if $u\in J(\mathrm{S}(K))$, then $uK=K$ and for each $x\in X$, $(xu)u=xu$, thus for each $p\in K$, $(xu)up=xup$, that is, for each $q\in K$, $(xu)q=xq$ and $(xu,x)\in\overline{\mathrm{P}}_A(X)=\Delta_X$, therefore for each $x\in X$, $xu=x$ and $u=e$.

Considering (ii), (iii) is a special case of (iv). The proof of (v) has a similar argument.

□

Theorem 1.4. *Let A be a nonempty subset of X, then:*

(a) (i) *(X,S) is distal if and only if $\mathrm{P}(X)=\Delta_X$,*

(ii) *(X,S) is A-distal if and only if $\mathrm{P}_A(X)=\Delta_X$,*

(iii) *if $A\in\overline{\mathcal{M}}(X,S)$, then (X,S) is $A\overset{(\overline{\mathrm{M}})}{\text{—}}$distal if and only if $\overline{\mathrm{P}}_A(X)=\Delta_X$,*

(iv) *if $A\in\overline{\overline{\mathcal{M}}}(X,S)$, then (X,S) is $A\overset{(\overline{\overline{\mathrm{M}}})}{\text{—}}$distal if and only if $\overline{\overline{\mathrm{P}}}_A(X)=\Delta_X$,*

(b) *if $(x,y)\in X\times X$, then:*

(i) *the following statements are equivalent:*

(1) $(x,y)\in\mathrm{P}(X)$,

(2) $\exists u\in J(\mathrm{E}(X))$, $xu=yu$,

(3) $\exists p\in\mathrm{E}(X)$, $xp=yp$,

(ii) *the following statements are equivalent:*
(1) $(x,y) \in \mathrm{P}_A(X)$,
(2) $\exists a \in A\ \exists u \in J(F(a,\mathrm{E}(X)))$, $xu = yu$,
(3) $\exists a \in A\ \ \exists p \in F(a,\mathrm{E}(X))$, $xp = yp$,

(iii) *if* $A \in \overline{\mathcal{M}}(X,S)$, *then the following statements are equivalent:*
(1) $(x,y) \in \overline{\mathrm{P}}_A(X)$,
(2) $\exists u \in J(F(A,\mathrm{E}(X)))$, $xu = yu$,
(3) $\exists p \in F(A,\mathrm{E}(X))$, $xp = yp$.

Proof. In each case for the sake of brevity we prove (iii).

(a) (iii) If (X,S) if $A\underline{\overset{(\overline{\mathrm{M}})}{\quad}}$distal, then $\overline{\mathrm{M}}(A) = \{\mathrm{E}(X)\}$, and if $(x,y) \in \overline{\mathrm{P}}_A(X)$, then for each $p \in \mathrm{E}(X)$, $xp = yp$, thus $x = xe = ye = y$ and $\overline{\mathrm{P}}_A(X) \subseteq \Delta_X$, therefore $\overline{\mathrm{P}}_A(X) = \Delta_X$. On the other hand, let $A \in \overline{\mathcal{M}}(X,S)$ and $\overline{\mathrm{P}}_A(X) = \Delta_X$, take $K \in \overline{\mathrm{M}}(A)$ and $u \in J(F(A,K))\,(\neq \varnothing)$, then $uK = K$ and for each $x \in X$ and $p \in K$, $xp = x(up) = (xu)p$, so $(x,xu) \in \overline{\mathrm{P}}_A(X) = \Delta_X$, that is, for each $x \in X$, $xu = x$ and $u = e$ so $K = \mathrm{E}(X)$, therefore (X,S) is $A\underline{\overset{(\overline{\mathrm{M}})}{\quad}}$distal.

(b) (iii) We have

$$\begin{aligned}
(1) &\Longrightarrow \exists K \in \overline{\mathrm{M}}(A)\ \ \forall p \in K,\ xp = yp \\
&\Longrightarrow \exists K \in \overline{\mathrm{M}}(A)\ \ \exists u \in \mathrm{J}(\mathrm{F}(A,K)),\ xu = yu\ \ (\text{since } A \in \overline{\mathcal{M}}(X,S)) \\
&\Longrightarrow (2), \\
(3) &\Longrightarrow \exists p \in \mathrm{F}(A,\mathrm{E}(X))\ \ \forall q \in p\mathrm{E}(X),\ xq = yq \\
&\Longrightarrow \exists p \in \mathrm{F}(A,\mathrm{E}(X))\ \ \exists L \in \overline{\mathrm{M}}(A)\ \ \forall q \in p\mathrm{E}(X), \\
&\qquad\quad (xq = yq \wedge L \subseteq p\mathrm{E}(X))\ \ (\text{by [4, Corollary 1.3]}) \\
&\Longrightarrow \exists L \in \overline{\mathrm{M}}(A)\ \ \forall q \in L,\ xq = yq \\
&\Longrightarrow (1).
\end{aligned} \tag{1.7}$$

□

Theorem 1.5. *Let A be a nonempty subset of X, then*

(a) (i) *the following statements are equivalent:*
(1) $\mathrm{Min}(\mathrm{E}(X))$ *is singleton,*
(2) $\mathrm{P}(X)$ *is a transitive relation on* X,
(3) $\mathrm{P}(X)$ *is an equivalence relation on* X,

(ii) *if* $A \in \overline{\mathcal{M}}(X,S)$, *then the following statements are equivalent:*
(1) $\overline{\mathrm{M}}(A)$ *is singleton,*
(2) $\overline{\mathrm{P}}_A(X)$ *is a transitive relation on* X,
(3) $\overline{\mathrm{P}}_A(X)$ *is an equivalence relation on* X,

(iii) *if* $A \in \overline{\overline{\mathcal{M}}}(X,S)$, *then the following statements are equivalent:*
(1) $\overline{\overline{\mathrm{M}}}(A)$ *is singleton,*
(2) $\overline{\overline{\mathrm{P}}}_A(X)$ *is a transitive relation on* X,
(3) $\overline{\overline{\mathrm{P}}}_A(X)$ *is an equivalence relation on* X,

(b) *if S is an abelian semigroup, then:*
(i) *if* $\mathrm{P}(X)$ *is a closed relation on* X, *then* $\mathrm{P}(X)$ *is an equivalence relation on* X,
(ii) *if* $A \in \overline{\mathcal{M}}(X,S)$ *and* $\overline{\mathrm{P}}_A(X)$ *is a closed relation on* X, *then* $\overline{\mathrm{P}}_A(X)$ *is an equivalence relation on* X,
(iii) *if* $A \in \overline{\overline{\mathcal{M}}}(X,S)$ *and* $\overline{\overline{\mathrm{P}}}_A(X)$ *is a closed relation on* X, *then* $\overline{\overline{\mathrm{P}}}_A(X)$ *is an equivalence relation on* X.

Proof. (a) (ii) By Note 1.3(a), it is enough to show that (3) implies (1). Let $\overline{\mathrm{P}}_A(X)$ be an equivalence relation on X, $K,L \in \overline{\mathrm{M}}(A)$ and $u \in J(F(A,K))$, there exists $v \in J(F(A,L))$ such that $uv = u$ and $vu = v$ [4, Theorem 7.1(a)], moreover, $u\mathrm{E}(X) = uK = K$, $v\mathrm{E}(X) = vL = L$ [4, Corollary 1.5(Table 3)], and for each $x \in X$, $p \in K$ and $q \in L$ we have: $(xu)p = x(up) = xp$ and $(xv)q = x(vq) = xq$. Therefore $(xu,x),(x,xv) \in \overline{\mathrm{P}}_A(X)$ and by the transitivity of $\overline{\mathrm{P}}_A(X)$, $(xu,xv) \in \overline{\mathrm{P}}_A(X)$, thus there exists $N \in \overline{\mathrm{M}}(A)$ such that for each $l \in N$, $xul = xvl$. We know there exists $w \in J(F(A,N))$, such that $uw = u$ and $vw = (vu)w = v(uw) = vu = v$ [4, Theorem 1.7(a)] thus $xu = xuw = xvw = xv$ (for each $x \in X$), so $u = v$ and $K = u\mathrm{E}(X) = v\mathrm{E}(X) = L$. Therefore $\overline{\mathrm{M}}(A)$ is singleton.

(b) (ii) Let $A \in \overline{\mathcal{M}}(X,S)$ and $\overline{\mathrm{P}_A}(X)$ be a closed relation on X, then for each $(x,y),(y,z) \in \overline{\mathrm{P}}_A(X)$, there exists $K \in \overline{\mathrm{M}}(A)$ such that for each $p \in K$, $xp = yp$. Let $u \in J(F(A,K))$ $(\neq \emptyset)$, then $xu = yu$. Now by Note 1.3(b), we have $(yu,zu) \in \overline{\mathrm{P}}_A(X)$. Choose $L \in \overline{\mathrm{M}}(A)$ such that for each $q \in L$, $yuq = zuq$. There exists $v \in J(F(A,L))$, such that $uv = u$ [4, Theorem 1.7(a)] thus $xu = yu = yuv = zuv = zu$, by Theorem 1.4(iii), $(x,z) \in \overline{\mathrm{P}}_A(X)$ and $\overline{\mathrm{P}}_A(X)$ is a transitive relation on X, thus by (a (ii)) $\overline{\mathrm{P}}_A(X)$ is an equivalence relation on X. □

Note 1.6. Let A be a nonempty subset of X, let $\varphi : (X,S) \to (Y,S)$ be an onto homomorphism. Define $\varphi\times\varphi : X\times X \to Y\times Y$ by $\varphi\times\varphi(x,y) = (\varphi(x),\varphi(y))$ $(\forall (x,y) \in X\times X)$, using Definition 1.1, we have

(a) if $K \in \overline{\mathrm{M}}(A)$, then there exists $L \in \overline{\mathrm{M}}(\varphi(A))$ such that $L \subseteq \hat{\varphi}(K)$,

(b) (i) $\varphi\times\varphi(\mathrm{P}(X)) \subseteq \mathrm{P}(Y)$,

(ii) $\varphi\times\varphi(\mathrm{P}_A(X)) \subseteq \mathrm{P}_{\varphi(A)}(Y)$,

(iii) $\varphi\times\varphi(\overline{\mathrm{P}}_A(X)) \subseteq \overline{\mathrm{P}}_{\varphi(A)}(Y)$,

(c) (i) $\mathrm{P}(X) \subseteq \bigcap_{\mathfrak{R}\in\mathfrak{I}} \mathfrak{R}$,

(ii) $\mathrm{P}_A(X) \subseteq \bigcap_{\mathfrak{R}\in\mathfrak{I}_0} \mathfrak{R}$,

(iii) $\overline{\mathrm{P}}_A(X) \subseteq \bigcap_{\mathfrak{R}\in\mathfrak{I}_1} \mathfrak{R}$.

Proof. (a) If $K \in \overline{\mathrm{M}}(A)$, then $\hat{\varphi}(K)$ is a closed right ideal of $\mathrm{E}(Y)$. On the other hand, for each $a \in A$, $aK = a\mathrm{E}(X)$ thus $\varphi(a)\hat{\varphi}(K) = \varphi(aK) = \varphi(a\mathrm{E}(X)) = \varphi(a)\hat{\varphi}(\mathrm{E}(X)) = \varphi(a)\mathrm{E}(Y)$, therefore there exists $L \in \overline{\mathrm{M}}(\varphi(A))$ such that $L \subseteq \hat{\varphi}(K)$ [4, Corollary 1.3(b)].

(b) Let $(x,y) \in X\times X$.

(ii) If $(x,y) \in \mathrm{P}_A(X)$, then there exists $a \in A$ and $K \in \mathrm{M}(a)$ such that for each $p \in K$, $xp = yp$ and $\varphi(x)\hat{\varphi}(p) = \varphi(y)\hat{\varphi}(p)$, by (a) there exists $L \in \mathrm{M}(\varphi(a))$ such that $L \subseteq \hat{\varphi}(K)$, so for each $q \in L$, $\varphi(x)q = \varphi(y)q$, therefore, $\varphi\times\varphi(x,y) = (\varphi(x),\varphi(y)) \in \mathrm{P}_{\varphi(A)}(Y)$.

(iii) If $(x,y) \in \overline{\mathrm{P}}_A(X)$, then there exists $K \in \overline{\mathrm{M}}(A)$ such that for each $p \in K$, $xp = yp$ and $\varphi(x)\hat{\varphi}(p) = \varphi(y)\hat{\varphi}(p)$, by (a) there exists $L \in \overline{\mathrm{M}}(\varphi(A))$ such that $L \subseteq \hat{\varphi}(K)$, so for each $q \in L$, $\varphi(x)q = \varphi(y)q$, therefore, $\varphi\times\varphi(x,y) = (\varphi(x),\varphi(y)) \in \overline{\mathrm{P}}_{\varphi(A)}(Y)$.

(c) (ii) Let $\mathfrak{R} \in \mathfrak{I}_0$, then

$$\begin{aligned}\mathfrak{R} \in \mathfrak{I}_0 &\Longrightarrow \left(\frac{X}{\mathfrak{R}},S\right) \text{ is } [A]_{\mathfrak{R}}\text{-distal}\\ &\Longrightarrow \mathrm{P}_{[A]_{\mathfrak{R}}}\left(\frac{X}{\mathfrak{R}}\right) = \Delta_{X/\mathfrak{R}}\end{aligned}$$

$$
\begin{aligned}
&\Longrightarrow \pi_{\Re}\times\pi_{\Re}(\mathrm{P}_A(X)) \subseteq \mathrm{P}_{\pi_{\Re}(A)}\left(\frac{X}{\Re}\right) = \mathrm{P}_{[A]_{\Re}}\left(\frac{X}{\Re}\right) = \Delta_{X/\Re}\\
&\Longrightarrow \pi_{\Re}\times\pi_{\Re}(\mathrm{P}_A(X)) = \Delta_{X/\Re}\\
&\Longrightarrow \forall (x,y)\in \mathrm{P}_A(X),\ [x]_{\Re} = [y]_{\Re}\\
&\Longrightarrow \forall (x,y)\in \mathrm{P}_A(X),\ (x,y)\in \Re\\
&\Longrightarrow \mathrm{P}_A(X) \subseteq \Re
\end{aligned}
\tag{1.8}
$$

so $\mathrm{P}_A(X) \subseteq \bigcap_{\Re\in\Im_0}\Re$. □

Definition 1.7. Let $\varphi : (X,S) \to (Y,S)$ be an onto homomorphism, $\mathrm{R}(\varphi) = \{(x,y) \in X\times X \mid \varphi(x) = \varphi(y)\}$, and let A be a nonempty subset of X, and let B be a nonempty subset of Y, then

(a) (Y,S) is a distal factor of (X,S) (under φ) if $\mathrm{R}(\varphi)\cap\mathrm{P}(X) = \Delta_X$,
(b) (Y,S) is an $A\underline{\overset{(-)}{}}$distal (or simply A-distal) factor of (X,S) (under φ) if $\mathrm{R}(\varphi)\cap \mathrm{P}_A(X) = \Delta_X$,
(c) (Y,S) is an $A\underline{\overset{(\overline{\mathrm{M}})}{}}$distal factor of (X,S) (under φ) if $\mathrm{R}(\varphi)\cap\overline{\mathrm{P}}_A(X) = \Delta_X$,
(d) (Y,S) is an $A\underline{\overset{(\overline{\overline{\mathrm{M}}})}{}}$distal factor of (X,S) (under φ) if $\mathrm{R}(\varphi)\cap\overline{\overline{\mathrm{P}}}_A(X) = \Delta_X$,
(e) (X,S) is a distal extension of (Y,S) (under φ) if $\mathrm{R}(\varphi)\cap\mathrm{P}(X) = \Delta_X$,
(f) (X,S) is a $B\underline{\overset{(-)}{}}$distal (or simply B-distal) extension of (Y,S) (under φ) if $\mathrm{R}(\varphi)\cap \mathrm{P}_{\varphi^{-1}(B)}(X) = \Delta_X$,
(g) (X,S) is a $B\underline{\overset{(\overline{\mathrm{M}})}{}}$distal extension of (Y,S) (under φ) if $\mathrm{R}(\varphi)\cap\overline{\mathrm{P}}_{\varphi^{-1}(B)}(X) = \Delta_X$,
(h) (X,S) is a $B\underline{\overset{(\overline{\overline{\mathrm{M}}})}{}}$distal extension of (Y,S) (under φ) if $\mathrm{R}(\varphi)\cap\overline{\overline{\mathrm{P}}}_{\varphi^{-1}(B)}(X) = \Delta_X$,
(a)′ (Y,S) is a proximal factor of (X,S) (under φ) if $\mathrm{R}(\varphi)\subseteq\mathrm{P}(X)$,
(b)′ (Y,S) is an $A\underline{\overset{(-)}{}}$proximal (or simply A-proximal) factor of (X,S) (under φ) if $\mathrm{R}(\varphi)\subseteq\mathrm{P}_A(X)$,
(c)′ (Y,S) is an $A\underline{\overset{(\overline{\mathrm{M}})}{}}$proximal factor of (X,S) (under φ) if $\mathrm{R}(\varphi)\subseteq\overline{\mathrm{P}}_A(X)$,
(d)′ (Y,S) is an $A\underline{\overset{(\overline{\overline{\mathrm{M}}})}{}}$proximal factor of (X,S) (under φ) if $\mathrm{R}(\varphi)\subseteq\overline{\overline{\mathrm{P}}}_A(X)$,
(e)′ (X,S) is a proximal extension of (Y,S) (under φ) if $\mathrm{R}(\varphi)\subseteq\mathrm{P}(X)$,
(f)′ (X,S) is a $B\underline{\overset{(-)}{}}$proximal (or simply B-proximal) extension of (Y,S) (under φ) if $\mathrm{R}(\varphi)\subseteq\mathrm{P}_{\varphi^{-1}(B)}(X)$,
(g)′ (X,S) is a $B\underline{\overset{(\overline{\mathrm{M}})}{}}$proximal extension of (Y,S) (under φ) if $\mathrm{R}(\varphi)\subseteq\overline{\mathrm{P}}_{\varphi^{-1}(B)}(X)$,
(h)′ (X,S) is a $B\underline{\overset{(\overline{\overline{\mathrm{M}}})}{}}$proximal extension of (Y,S) (under φ) if $\mathrm{R}(\varphi)\subseteq\overline{\overline{\mathrm{P}}}_{\varphi^{-1}(B)}(X)$.

Theorem 1.8. *Let $\varphi : (X,S) \to (Y,S)$ be an onto homomorphism, let A be a nonempty subset of X, let B be a nonempty subset of Y, and consider the following statements:*

(π_1) *(Y,S) is a distal factor of (X,S) under φ,*
(π_2) *(Y,S) is an A-distal factor of (X,S) under φ,*
(π_3) *(Y,S) is an $A\underline{\overset{(\overline{\mathrm{M}})}{}}$distal factor of (X,S) under φ,*
(π_4) *(Y,S) is an $A\underline{\overset{(\overline{\overline{\mathrm{M}}})}{}}$distal factor of (X,S) under φ (by the assumption $\overline{\overline{\mathrm{M}}}(A)\neq\emptyset$),*
(ρ_1) *(X,S) is a distal extension of (Y,S) under φ,*
(ρ_2) *(X,S) is a B-distal extension of (Y,S) under φ,*
(ρ_3) *(X,S) is a $B\underline{\overset{(\overline{\mathrm{M}})}{}}$distal extension of (Y,S) under φ,*
(ρ_4) *(X,S) is a $B\underline{\overset{(\overline{\overline{\mathrm{M}}})}{}}$distal extension of (Y,S) under φ (by the assumption $\overline{\overline{\mathrm{M}}}(\varphi^{-1}(B))\neq\emptyset$),*

(π'_1) (Y,S) *is a proximal factor of* (X,S) *under* φ,
(π'_2) (Y,S) *is an* A*-proximal factor of* (X,S) *under* φ,
(π'_3) (Y,S) *is an* $A\frac{(\overline{\mathrm{M}})}{}$*proximal factor of* (X,S) *under* φ,
(π'_4) (Y,S) *is an* $A\frac{(\overline{\overline{\mathrm{M}}})}{}$*proximal factor of* (X,S) *under* φ *(by the assumption* $\overline{\overline{\mathrm{M}}}(A) \neq \emptyset$*)*,
(ρ'_1) (X,S) *is a proximal extension of* (Y,S) *under* φ,
(ρ'_2) (X,S) *is a* B*-proximal extension of* (Y,S) *under* φ,
(ρ'_3) (X,S) *is a* $B\frac{(\overline{\mathrm{M}})}{}$*proximal extension of* (Y,S) *under* φ,
(ρ'_4) (X,S) *is a* $B\frac{(\overline{\overline{\mathrm{M}}})}{}$*proximal extension of* (Y,S) *under* φ *(by the assumption* $\overline{\overline{\mathrm{M}}}(\varphi^{-1}(B)) \neq \emptyset$*)*,

then we have the following tables:

TABLE 1.1. The mark "$\surd$" indicates that for the corresponding case we have: "$(\pi_i \Rightarrow \pi_j) \wedge (\rho_i \Rightarrow \rho_j)$"

$i \backslash j$	1	2	3	4
1	$\surd$	$\surd$	$\surd$	$\surd$
2		$\surd$	$\surd$	
3			$\surd$	
4				$\surd$

TABLE 1.2. The mark "$\surd$" indicates that for the corresponding case we have: "$(\pi'_i \Rightarrow \pi'_j) \wedge (\rho'_i \Rightarrow \rho'_j)$"

$i \backslash j$	1	2	3	4
1	$\surd$			
2	$\surd$	$\surd$		
3	$\surd$	$\surd$	$\surd$	
4	$\surd$			$\surd$

PROOF. We have the following conditional statements:

$$
\begin{aligned}
(\pi_1) &\Longrightarrow \mathrm{R}(\varphi) \cap \mathrm{P}(X) = \Delta_X \\
&\Longrightarrow \big(\mathrm{R}(\varphi) \cap \overline{\mathrm{P}}_A(X) \subseteq \mathrm{R}(\varphi) \cap \mathrm{P}_A(X) \subseteq \mathrm{R}(\varphi) \cap \mathrm{P}(X) = \Delta_X \\
&\qquad \wedge \mathrm{R}(\varphi) \cap \overline{\overline{\mathrm{P}}}_A(X) \subseteq \mathrm{R}(\varphi) \cap \mathrm{P}(X) = \Delta_X\big) \ \text{(by Note 1.3(c))} \\
&\Longrightarrow \big(\mathrm{R}(\varphi) \cap \overline{\mathrm{P}}_A(X) = \mathrm{R}(\varphi) \cap \mathrm{P}_A(X) = \Delta_X \wedge \mathrm{R}(\varphi) \cap \overline{\overline{\mathrm{P}}}_A(X) \subseteq \Delta_X\big) \\
&\Longrightarrow (\pi_2 \wedge \pi_3 \wedge \pi_4), \\
(\rho_1) &\Longrightarrow \mathrm{R}(\varphi) \cap \mathrm{P}(X) = \Delta_X \\
&\Longrightarrow \big(\mathrm{R}(\varphi) \cap \overline{\mathrm{P}}_{\varphi^{-1}(B)}(X) \subseteq \mathrm{R}(\varphi) \cap \mathrm{P}_{\varphi^{-1}(B)}(X) \subseteq \mathrm{R}(\varphi) \cap \mathrm{P}(X) = \Delta_X \\
&\qquad \wedge \mathrm{R}(\varphi) \cap \overline{\overline{\mathrm{P}}}_{\varphi^{-1}(B)}(X) \subseteq \mathrm{R}(\varphi) \cap \mathrm{P}(X) = \Delta_X\big) \ \text{(by Note 1.3(c))} \\
&\Longrightarrow \big(\mathrm{R}(\varphi) \cap \overline{\mathrm{P}}_{\varphi^{-1}(B)}(X) = \mathrm{R}(\varphi) \cap \mathrm{P}_{\varphi^{-1}(B)}(X) = \Delta_X \\
&\qquad \wedge \mathrm{R}(\varphi) \cap \overline{\overline{\mathrm{P}}}_{\varphi^{-1}(B)}(X) \subseteq \Delta_X\big)
\end{aligned}
$$

$$
\begin{aligned}
&\Longrightarrow (\rho_2 \wedge \rho_3 \wedge \rho_4),\\
(\pi_2) &\Longrightarrow \mathrm{R}(\varphi) \cap \mathrm{P}_A(X) = \Delta_X\\
&\Longrightarrow \mathrm{R}(\varphi) \cap \overline{\mathrm{P}}_A(X) \subseteq \mathrm{R}(\varphi) \cap \mathrm{P}_A(X) = \Delta_X \ \text{(by Note 1.3(c))}\\
&\Longrightarrow \mathrm{R}(\varphi) \cap \overline{\mathrm{P}}_A(X) = \Delta_X\\
&\Longrightarrow (\pi_3),\\
(\rho_2) &\Longrightarrow \mathrm{R}(\varphi) \cap \mathrm{P}_{\varphi^{-1}(B)}(X) = \Delta_X\\
&\Longrightarrow \mathrm{R}(\varphi) \cap \overline{\mathrm{P}}_{\varphi^{-1}(B)}(X) \subseteq \mathrm{R}(\varphi) \cap \mathrm{P}_{\varphi^{-1}(B)}(X) = \Delta_X \ \text{(by Note 1.3(c))}\\
&\Longrightarrow \mathrm{R}(\varphi) \cap \overline{\mathrm{P}}_{\varphi^{-1}(B)}(X) = \Delta_X\\
&\Longrightarrow (\rho_3)
\end{aligned}
\tag{1.9}
$$

these complete the proof of Table 1.1, also

$$
\begin{aligned}
(\pi_3') &\Longrightarrow \mathrm{R}(\varphi) \subseteq \overline{\mathrm{P}}_A(X)\\
&\Longrightarrow \mathrm{R}(\varphi) \subseteq \overline{\mathrm{P}}_A(X) \subseteq \mathrm{P}_A(X) \ \text{(by Note 1.3(c))}\\
&\Longrightarrow (\pi_2')\\
&\Longrightarrow \mathrm{R}(\varphi) \subseteq \mathrm{P}_A(X) \subseteq \mathrm{P}(X) \ \text{(by Note 1.3(c))}\\
&\Longrightarrow (\pi_1'),\\
(\rho_3') &\Longrightarrow \mathrm{R}(\varphi) \subseteq \overline{\mathrm{P}}_{\varphi^{-1}(B)}(X)\\
&\Longrightarrow \mathrm{R}(\varphi) \subseteq \overline{\mathrm{P}}_{\varphi^{-1}(B)}(X) \subseteq \mathrm{P}_{\varphi^{-1}(B)}(X) \ \text{(by Note 1.3(c))}\\
&\Longrightarrow (\rho_2')\\
&\Longrightarrow \mathrm{R}(\varphi) \subseteq \mathrm{P}_{\varphi^{-1}(B)}(X) \subseteq \mathrm{P}(X) \ \text{(by Note 1.3(c))}\\
&\Longrightarrow (\rho_1'),\\
(\pi_4') &\Longrightarrow \mathrm{R}(\varphi) \subseteq \overline{\overline{\mathrm{P}}}_A(X)\\
&\Longrightarrow \mathrm{R}(\varphi) \subseteq \overline{\overline{\mathrm{P}}}_A(X) \subseteq \mathrm{P}(X) \ \text{(by Note 1.3(c))}\\
&\Longrightarrow (\pi_1'),\\
(\rho_4') &\Longrightarrow \mathrm{R}(\varphi) \subseteq \overline{\overline{\mathrm{P}}}_{\varphi^{-1}(B)}(X)\\
&\Longrightarrow \mathrm{R}(\varphi) \subseteq \overline{\overline{\mathrm{P}}}_{\varphi^{-1}(B)}(X) \subseteq \mathrm{P}(X) \ \text{(by Note 1.3(c))}\\
&\Longrightarrow (\rho_1'),
\end{aligned}
\tag{1.10}
$$

□

these complete the proof of Table 1.2.

Theorem 1.9. *Let $\varphi : (X,S) \to (Y,S)$ be an onto homomorphism and $\varnothing \neq C \subseteq A \subseteq X$, and $\varnothing \neq D \subseteq B \subseteq Y$, then*

(a) *if (Y,S) is an A-distal factor of (X,S), then (Y,S) is a C-distal factor of (X,S),*
(b) *if (Y,S) is a $C\underline{\text{(M)}}$distal factor of (X,S), then (Y,S) is an $A\underline{\text{(M)}}$distal factor of (X,S),*
(c) *if (X,S) is a B-distal extension of (Y,S), then (X,S) is a D-distal extension of (Y,S),*
(d) *if (X,S) is a $D\underline{\text{(M)}}$distal extension of (Y,S), then (X,S) is a $B\underline{\text{(M)}}$distal extension of (Y,S),*

(e) *if* (Y,S) *is a* C*-proximal factor of* (X,S)*, then* (Y,S) *is an* A*-proximal factor of* (X,S)*,*

(f) *if* (Y,S) *is an* $A\overset{(\mathrm{M})}{\text{—}}$*proximal factor of* (X,S)*, then* (Y,S) *is a* $C\overset{(\mathrm{M})}{\text{—}}$*proximal factor of* (X,S)

(g) *if* (X,S) *is a* D*-proximal extension of* (Y,S)*, then* (X,S) *is a* B*-proximal extension of* (Y,S)*,*

(h) *if* (X,S) *is a* $B\overset{(\mathrm{M})}{\text{—}}$*proximal extension of* (Y,S)*, then* (X,S) *is a* $D\overset{(\mathrm{M})}{\text{—}}$*proximal extension of* (Y,S)*,*

the factors and extensions are under φ.

Proof. (a) (Y,S) is an A-distal factor of (X,S)

$$\begin{aligned}
&\Longrightarrow \mathrm{R}(\varphi)\cap\mathrm{P}_A(X)=\Delta_X\\
&\Longrightarrow \mathrm{R}(\varphi)\cap\mathrm{P}_C(X)\subseteq\mathrm{R}(\varphi)\cap\mathrm{P}_A(X)=\Delta_X \ \text{(by Note 1.3(c))}\\
&\Longrightarrow \mathrm{R}(\varphi)\cap\mathrm{P}_C(X)=\Delta_X\\
&\Longrightarrow (Y,S)\ \text{is a } C\text{-distal factor of } (X,S),
\end{aligned}\tag{1.11}$$

(b) (Y,S) is a $C\overset{(\mathrm{M})}{\text{—}}$distal factor of (X,S)

$$\begin{aligned}
&\Longrightarrow \mathrm{R}(\varphi)\cap\overline{\mathrm{P}}_C(X)=\Delta_X\\
&\Longrightarrow \mathrm{R}(\varphi)\cap\overline{\mathrm{P}}_A(X)\subseteq\mathrm{R}(\varphi)\cap\overline{\mathrm{P}}_C(X)=\Delta_X \ \text{(by Note 1.3(c))}\\
&\Longrightarrow \mathrm{R}(\varphi)\cap\overline{\mathrm{P}}_A(X)=\Delta_X\\
&\Longrightarrow (Y,S)\ \text{is an } A\overset{(\mathrm{M})}{\text{—}}\ \text{distal factor of } (X,S),
\end{aligned}\tag{1.12}$$

(e) (Y,S) is a C-proximal factor of (X,S)

$$\begin{aligned}
&\Longrightarrow \mathrm{R}(\varphi)\subseteq\mathrm{P}_C(X)\\
&\Longrightarrow \mathrm{R}(\varphi)\subseteq\mathrm{P}_C(X)\subseteq\mathrm{P}_A(X)\ \text{(by Note 1.3(c))}\\
&\Longrightarrow (Y,S)\ \text{is an } A\text{-proximal factor of } (X,S),
\end{aligned}\tag{1.13}$$

(f) (Y,S) is an $A\overset{(\mathrm{M})}{\text{—}}$proximal factor of (X,S)

$$\begin{aligned}
&\Longrightarrow \mathrm{R}(\varphi)\subseteq\overline{\mathrm{P}}_A(X)\\
&\Longrightarrow \mathrm{R}(\varphi)\subseteq\overline{\mathrm{P}}_A(X)\subseteq\overline{\mathrm{P}}_C(X)\ \text{(by Note 1.3(c))}\\
&\Longrightarrow (Y,S)\ \text{is a } C\overset{(\mathrm{M})}{\text{—}}\ \text{proximal factor of } (X,S).
\end{aligned}\tag{1.14}$$

□

Theorem 1.10 (associative and inheritance laws). *Let* $\varphi:(X,S)\to(Y,S)$ *and* $\psi:(Y,S)\to(Z,S)$ *be two onto homomorphisms, and let* A *be a nonempty subset of* X*, and* B *be a nonempty subset of* Y*, then we have*

(a) Associative laws.

(i) *(((*Z,S*) is a distal factor of* (Y,S) *(under* ψ*))* $\wedge$ *((*Y,S*) is a distal factor of* (X,S) *(under* φ*)))* $\Rightarrow$ *((*Z,S*) is a distal factor of* (X,S) *(under* $\psi\circ\varphi$*)),*

(ii) *(((*Z,S*) is a* $\varphi(A)$*-distal factor of* (Y,S) *(under* ψ*))* $\wedge$ *((*Y,S*) is an* A*-distal factor of* (X,S) *(under* φ*)))* $\Rightarrow$ *((*Z,S*) is an* A*-distal factor of* (X,S) *(under* $\psi\circ\varphi$*)),*

(iii) *$(((Z,S)$ is a $\varphi(A)\overset{(\overline{M})}{\text{—}}$distal factor of (Y,S) (under ψ)) $\wedge$ $((Y,S)$ is an $A\overset{(\overline{M})}{\text{—}}$distal factor of (X,S) (under φ))) $\Rightarrow$ $((Z,S)$ is an $A\overset{(\overline{M})}{\text{—}}$distal factor of (X,S) (under $\psi\circ\varphi$)),*

(i)′ *$(((X,S)$ is a distal extension of (Y,S) (under φ)) $\wedge$ $((Y,S)$ is a distal extension of (Z,S) (under ψ))) $\Rightarrow$ $((X,S)$ is a distal extension of (Z,S) (under $\psi\circ\varphi$)),*

(ii)′ *$(((X,S)$ is a $\psi^{-1}(B)$-distal extension of (Y,S)(under φ)) $\wedge$ $((Y,S)$ is a B-distal extension of (Z,S) (under ψ))) $\Rightarrow$ $((X,S)$ is a B-distal extension of (Z,S) (under $\psi\circ\varphi$)),*

(iii)′ *$(((X,S)$ is a $\psi^{-1}(B)\overset{(\overline{M})}{\text{—}}$distal extension of (Y,S) (under φ)) $\wedge$ $((Y,S)$ is a $B\overset{(\overline{M})}{\text{—}}$distal extension of (Z,S) (under ψ))) $\Rightarrow$ $((X,S)$ is a $B\overset{(\overline{M})}{\text{—}}$distal extension of (Z,S) (under $\psi\circ\varphi$)),*

(b) Inheritance laws.

(i) *$((Z,S)$ is a distal factor of (X,S) (under $\psi\circ\varphi$)) $\Rightarrow$ $((Y,S)$ is a distal factor of (X,S) (under φ)),*

(ii) *$((Z,S)$ is an A-distal factor of (X,S) (under $\psi\circ\varphi$)) $\Rightarrow$ $((Y,S)$ is an A-distal factor of (X,S) (under φ)),*

(iii) *$((Z,S)$ is an $A\overset{(\overline{M})}{\text{—}}$distal factor of (X,S) (under $\psi\circ\varphi$)) $\Rightarrow$ $((Y,S)$ is an $A\overset{(\overline{M})}{\text{—}}$distal factor of (X,S) (under φ)),*

(iv) *$((Z,S)$ is an $A\overset{(\overline{\overline{M}})}{\text{—}}$distal factor of (X,S) (under $\psi\circ\varphi$)) $\Rightarrow$ $((Y,S)$ is an $A\overset{(\overline{\overline{M}})}{\text{—}}$distal factor of (X,S) (under φ)),*

(v) *$((Z,S)$ is a proximal factor of (X,S) (under $\psi\circ\varphi$)) $\Rightarrow$ $((Y,S)$ is a proximal factor of (X,S) (under φ)),*

(vi) *$((Z,S)$ is an A-proximal factor of (X,S) (under $\psi\circ\varphi$)) $\Rightarrow$ $((Y,S)$ is an A-proximal factor of (X,S) (under φ)),*

(vii) *$((Z,S)$ is an $A\overset{(\overline{M})}{\text{—}}$proximal factor of (X,S) (under $\psi\circ\varphi$)) $\Rightarrow$ $((Y,S)$ is an $A\overset{(\overline{M})}{\text{—}}$proximal factor of (X,S) (under φ)),*

(vii) *$((Z,S)$ is an $A\overset{(\overline{\overline{M}})}{\text{—}}$proximal factor of (X,S) (under $\psi\circ\varphi$)) $\Rightarrow$ $((Y,S)$ is an $A\overset{(\overline{\overline{M}})}{\text{—}}$proximal factor of (X,S) (under φ)),*

(i)′ *$((X,S)$ is a distal extension of (Z,S) (under $\psi\circ\varphi$)) $\Rightarrow$ $((X,S)$ is a distal extension of (Y,S) (under φ)),*

(ii)′ *$((X,S)$ is a B-distal extension of (Z,S) (under $\psi\circ\varphi$)) $\Rightarrow$ $((X,S)$ is a $\psi^{-1}(B)$-distal extension of (Y,S) (under φ)),*

(iii)′ *$((X,S)$ is a $B\overset{(\overline{M})}{\text{—}}$distal extension of (Z,S) (under $\psi\circ\varphi$)) $\Rightarrow$ $((X,S)$ is a $\psi^{-1}(B)\overset{(\overline{M})}{\text{—}}$distal extension of (Y,S) (under φ)),*

(iv)′ *$((X,S)$ is a $B\overset{(\overline{\overline{M}})}{\text{—}}$distal extension of (Z,S) (under $\psi\circ\varphi$)) $\Rightarrow$ $((X,S)$ is a $\psi^{-1}(B)\overset{(\overline{\overline{M}})}{\text{—}}$distal extension of (Y,S) (under φ)),*

(v)′ *$((X,S)$ is a proximal extension of (Z,S) (under $\psi\circ\varphi$)) $\Rightarrow$ $((X,S)$ is a proximal extension of (Y,S) (under φ)),*

(vi)′ *$((X,S)$ is a B-proximal extension of (Z,S) (under $\psi\circ\varphi$)) $\Rightarrow$ $((X,S)$ is a $\psi^{-1}(B)$-proximal extension of (Y,S) (under φ)),*

(vii)′ *$((X,S)$ is a $B\overset{(\overline{M})}{\text{—}}$proximal extension of (Z,S) (under $\psi\circ\varphi$)) $\Rightarrow$ $((X,S)$ is a $\psi^{-1}(B)\overset{(\overline{M})}{\text{—}}$proximal extension of (Y,S) (under φ)),* (viii)′ *$((X,S)$ is a $B\underline{\overset{(\overline{\overline{M}})}{}}$proximal extension of (Z,S) (under $\psi\circ\varphi$)) $\Rightarrow$ ((X,S) is a $\psi^{-1}(B)\underline{\overset{(\overline{\overline{M}})}{}}$proximal extension of (Y,S) (under φ)).*

Proof. (a) (ii) Let (Z,S) be a $\varphi(A)$-distal factor of (Y,S) under ψ, and let (Y,S) be an A-distal factor of (X,S) under φ, then $\mathrm{R}(\psi)\cap \mathrm{P}_{\varphi(A)}(Y)=\Delta_Y$ and $\mathrm{R}(\varphi)\cap\mathrm{P}_A(X)=\Delta_X$. Moreover, using the symbols of Note 1.6, we have $\varphi\times\varphi(\mathrm{R}(\psi\circ\varphi))\subseteq \mathrm{R}(\psi)$ so $\varphi\times\varphi(\mathrm{R}(\psi\circ\varphi)\cap\mathrm{P}_A(X))\subseteq\mathrm{R}(\psi)\cap\mathrm{P}_{\varphi(A)}(Y)=\Delta_Y$, thus $\varphi\times\varphi(\mathrm{R}(\psi\circ\varphi)\cap\mathrm{P}_A(X))=\Delta_Y$, that is, $\mathrm{R}(\psi\circ\varphi)\cap\mathrm{P}_A(X)\subseteq\mathrm{R}(\varphi)$, thus $\mathrm{R}(\psi\circ\varphi)\cap\mathrm{P}_A(X)\subseteq\mathrm{R}(\varphi)\cap\mathrm{P}_A(X)$, therefore $\mathrm{R}(\psi\circ\varphi)\cap\mathrm{P}_A(X)=\Delta_X$ and (Z,S) is an A-distal factor of (X,S) (under $\psi\circ\varphi$).

(b) Use $\mathrm{R}(\varphi)\subseteq\mathrm{R}(\psi\circ\varphi)$. □

Theorem 1.11. *Let B be a nonempty subset of X, let $\Sigma=\{\varphi_\alpha\mid\alpha\in\Gamma\}$ be a nonempty collection of the extensions of (X,S), $\alpha_0\in\Gamma$, $\times_\Sigma X_\alpha=\{(x_\alpha)_{\alpha\in\Gamma}\in\prod_{\alpha\in\Gamma}X_\alpha\mid\forall\alpha\in\Gamma\ \varphi_\alpha(x_\alpha)=\varphi_{\alpha_0}(x_{\alpha_0})\}$, for each $\gamma\in\Gamma$ let $\pi_\gamma:\times_\Sigma X_\alpha\to X_\gamma$ be the projection map on the γth coordinate, and $\varphi:\times_\Sigma X_\alpha\to X$ be such that $\varphi((x_\alpha)_{\alpha\in\Gamma})=\varphi_{\alpha_0}(x_{\alpha_0})$, then*

(a) *for each $\gamma\in\Gamma$, the following diagram commutes:*

$$\begin{tikzcd}(\times_\Sigma X_\alpha,S) \arrow[r, "\pi_\delta"] \arrow[d, "\varphi"'] & (X_\gamma,S) \arrow[dl, "\varphi_\gamma"] \\ (X,S) & \end{tikzcd} \tag{1.15}$$

(b) (i) *if for each $\alpha\in\Gamma$, (X_α,S) is a distal extension of (X,S) (under φ_α), then $(\times_\Sigma X_\alpha,S)$ is a distal extension of (X,S) (under φ),*

(ii) *if for each $\alpha\in\Gamma$, (X_α,S) is a B-distal extension of (X,S) (under φ_α), then $(\times_\Sigma X_\alpha,S)$ is a B-distal extension of (X,S) (under φ),*

(iii) *if for each $\alpha\in\Gamma$, (X_α,S) is a $B\underline{\overset{(\mathrm{M})}{}}$distal extension of (X,S) (under φ_α), then $(\times_\Sigma X_\alpha,S)$ is a $B\underline{\overset{(\mathrm{M})}{}}$distal extension of (X,S) (under φ).*

Proof. (b) (ii) By the definition of $\times_\Sigma X_\alpha$, we have

$$\begin{aligned}\mathrm{R}(\varphi)&=\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in(\times_\Sigma X_\alpha)\times(\times_\Sigma X_\alpha)\mid\forall\alpha\in\Gamma(x_\alpha,y_\alpha)\in\mathrm{R}(\varphi_\alpha)\}\\&=\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in(\times_\Sigma X_\alpha)\times(\times_\Sigma X_\alpha)\mid\exists\alpha\in\Gamma(x_\alpha,y_\alpha)\in\mathrm{R}(\varphi_\alpha)\},\end{aligned}\tag{1.16}$$

moreover, for each $((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in\mathrm{P}_{\varphi^{-1}(B)}(\times_\Sigma X_\alpha)$ and $\gamma\in\Gamma$, we have $(x_\gamma,y_\gamma)\in\mathrm{P}_{\varphi_\gamma^{-1}(B)}(X_\gamma)$, so if $((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in\mathrm{R}(\varphi)\cap\mathrm{P}_{\varphi^{-1}(B)}(\times_\Sigma X_\alpha)$,then $(x_\gamma,y_\gamma)\in\mathrm{R}(\varphi_\gamma)\cap\mathrm{P}_{\varphi_\gamma^{-1}(B)}(X_\gamma)$, this will give the desired result, that is, if for each $\alpha\in\Gamma$, $\mathrm{R}(\varphi_\alpha)\cap\mathrm{P}_{\varphi_\alpha^{-1}(B)}(X_\alpha)=\Delta_{X_\alpha}$, then $\mathrm{R}(\varphi)\cap\mathrm{P}_{\varphi^{-1}(B)}(\times_\Sigma X_\alpha)=(\Delta\times_\Sigma X_\alpha)$. □

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