OSCILLATION OF NONLINEAR DELAY DIFFERENCE EQUATIONS

JIANCHU JIANG

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ABSTRACT. We obtain some oscillation criteria for solutions of the nonlinear delay difference equation of the form $x_{n+1} - x_n + p_n \prod_{j=1}^m x_{n-k_j}^{\alpha_j} = 0$.

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1. Introduction. Consider the nonlinear delay difference equation

$$x_{n+1} - x_n + p_n \prod_{j=1}^m x_{n-k_j}^{\alpha_j} = 0, \quad n = 0, 1, 2, \dots, \quad (1.1)$$

where $p_n \geq 0$, $n = 0, 1, 2, \dots$, $0 \leq k_1 \leq k_2 \leq \dots \leq k_m$ are integers, $\alpha_j > 0$ are rational numbers with denominator of positive odd integers for each $j = 1, 2, \dots, m$ and $\sum_{j=1}^m \alpha_j = 1$.

Equation (1.1) is a discrete analogue of the following first-order nonlinear delay differential equation

$$x'(t) + p(t) \prod_{j=1}^m [x(t - \tau_j)]^{\alpha_j} = 0, \quad (1.2)$$

where $p(t) \in C([t_0, \infty), [0, \infty))$, $0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_m$, and α_j are the same as in (1.1). For (1.2), the oscillation of its solutions has been extensively studied in the literature, see, for example [2, 4, 11, 12].

When $k_1 = k_2 = \dots = k_m = k$, (1.1) reduces to the linear delay difference equation

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, \dots. \quad (1.3)$$

Recently, there has been a lot of activity concerning the oscillatory behavior of (1.3). See, for example, [1, 3, 5, 6, 7, 8, 9, 10]. In particular, [7] proved that every solution of (1.3) is oscillatory provided

$$\sum_{n=0}^{\infty} \left[\sum_{i=n}^{n+k} p_i \ln \left(\sum_{i=n}^{n+k} p_i + 1 - \text{sign} \sum_{i=n}^{n+k} p_i \right) - \sum_{i=n+1}^{n+k} p_i \ln \left(\sum_{i=n+1}^{n+k} p_i + 1 - \text{sign} \sum_{i=n+1}^{n+k} p_i \right) \right] = \infty. \quad (1.4)$$

Condition (1.4) improves many previous well-known results. Furthermore, (1.4) fits the case when $\sum_{i=n-k}^{n-1} p_i - (k/(k+1))^{k+1}$ oscillate or $\sum_{i=n-k}^{n-1} p_i \leq (k/(k+1))^{k+1}$.

Our main aim in this note is to generalize condition (1.4) to (1.1).

2. Main results

THEOREM 2.1. Assume that

$$\sum_{n=0}^{\infty} \left\{ \sum_{j=1}^m \alpha_j \sum_{i=n}^{n+k_j} p_i \ln \left[\sum_{j=1}^m \alpha_j \sum_{i=n}^{n+k_j} p_i + 1 - \operatorname{sign} \left(\sum_{j=1}^m \alpha_j \sum_{i=n}^{n+k_j} p_i \right) \right] \right. \\ \left. - \sum_{j=1}^m \alpha_j \sum_{i=n+1}^{n+k_j} p_i \ln \left[\sum_{j=1}^m \alpha_j \sum_{i=n+1}^{n+k_j} p_i + 1 - \operatorname{sign} \left(\sum_{j=1}^m \alpha_j \sum_{i=n+1}^{n+k_j} p_i \right) \right] \right\} = \infty. \quad (2.1)$$

Then every solution of (1.1) oscillates.

PROOF. Assume, by way of contradiction, that (1.1) has an eventually positive solutions $\{x_n\}$. Then there exists an integer $n_1 > 0$ such that $x_{n-k_m} > 0$, $x_{n+1} \leq x_n$, $n \geq n_1$. We define the functions $p(t)$ and $x(t)$ as follows:

$$p(t) = p_n, \quad x(t) = x_n + (t - n)(x_{n+1} - x_n), \quad n \leq t < n + 1, \quad n = 0, 1, 2, \dots \quad (2.2)$$

Let $x'(t)$ denote derivation on the right. Then

$$\begin{aligned} x(t) &> 0, \quad x'(t) \leq 0, \quad t \geq n_1, \\ x'(t) &= x_{n+1} - x_n \quad \text{for } n \leq t < n + 1, \quad n = 0, 1, 2, \dots \end{aligned} \quad (2.3)$$

Hence, (1.1) can be rewritten as

$$x'(t) + p(t) \prod_{j=1}^m (x[t - k_j])^{\alpha_j} = 0, \quad t \geq 0, \quad (2.4)$$

where here and in the sequel, $[\cdot]$ denotes the greatest integer function.

Set $\lambda(t) = -x'(t)/x(t)$ for $t \geq n_1$. Then $\lambda(t) \geq 0$ for $t \geq n_1$, and from (2.4) we have

$$\lambda(t) = p(t) \exp \left(\sum_{j=1}^m \alpha_j \int_{[t-k_j]}^t \lambda(s) ds \right), \quad t \geq n_1 + k_m, \quad (2.5)$$

or

$$\begin{aligned} \lambda(t) &\sum_{j=1}^m \alpha_j \int_t^{[t+k_j+1]} p(s) ds \\ &\geq p(t) \left(\sum_{j=1}^m \alpha_j \int_t^{[t+k_j+1]} \lambda(s) ds \right) \exp \left(\sum_{j=1}^m \alpha_j \int_{[t-k_j]}^t \lambda(s) ds \right), \quad t \geq n_1 + k_m. \end{aligned} \quad (2.6)$$

One can easily show that

$$\phi(r)e^x \geq \phi(r)x + \phi(r) \ln(er + 1 - \operatorname{sign} r), \quad r \geq 0, \quad x \geq R, \quad (2.7)$$

where $\phi(0) = 0$ and $\phi(r) \geq 0$ for $r > 0$. By the definition of $p(t)$, we see that $p(t)$ is

nonnegative and right-continuous. Therefore, it follows from

$$\sum_{j=1}^m \alpha_j \int_t^{[t+k_j+1]} p(s) ds = 0 \quad (2.8)$$

that $p(t) = 0$. Applying inequalities (2.7) to the right side of (2.6), we obtain

$$\lambda(t) \sum_{j=1}^m \alpha_j \int_t^{[t+k_j+1]} p(s) ds \geq p(t) \sum_{j=1}^m \alpha_j \int_{[t-k_j]}^t \lambda(s) ds + p(t) \ln A(t), \quad n \geq n_1 + k_m, \quad (2.9)$$

where

$$A(t) = e \sum_{j=1}^m \alpha_j \int_t^{[t+k_j+1]} p(s) ds + 1 - \text{sign} \left(\sum_{j=1}^m \alpha_j \int_t^{[t+k_j+1]} p(s) ds \right). \quad (2.10)$$

Set $n_2 = n_1 + k_m$. Integrating both sides of (2.9) from n_2 to $N > n_2 + 2k_m$, we have

$$\begin{aligned} & \sum_{j=1}^m \alpha_j \int_{n_2}^N \lambda(t) \int_t^{[t+k_j+1]} p(s) ds dt \\ & \geq \sum_{j=1}^m \alpha_j \int_{n_2}^N p(t) \int_{[t-k_j]}^t \lambda(s) ds dt + \int_{n_2}^N p(t) \ln A(t) dt. \end{aligned} \quad (2.11)$$

Interchanging the order of integration, we get

$$\begin{aligned} \int_{n_2}^N p(t) \int_{[t-k_j]}^t \lambda(s) ds dt & \geq \int_{n_2}^{N-k_j} \lambda(s) \int_s^{[s+k_j+1]} p(t) dt ds \\ & = \int_{n_2}^{N-k_j} \lambda(t) \int_t^{[t+k_j+1]} p(s) ds dt. \end{aligned} \quad (2.12)$$

Substituting this into (2.11), we have

$$\sum_{j=1}^m \alpha_j \int_{N-k_j}^N \lambda(t) \int_t^{[t+k_j+1]} p(s) ds dt \geq \int_{n_2}^N p(t) \ln A(t) dt. \quad (2.13)$$

From (1.1) we have

$$x_{n+1} - x_n + p_n x_n^{1-\alpha_m} x_{n-k_m}^{\alpha_m} \leq 0, \quad n \geq n_1. \quad (2.14)$$

Set $y_n = x_n^{\alpha_m}$ for $n \geq n_1$. Then

$$y_{n+1} - y_n + \alpha_m p_n y_{n-k_m} \leq 0, \quad n \geq n_1 + k_m. \quad (2.15)$$

It follows that $\alpha_m \sum_{i=n-k_m}^n p_i \leq 1$, and so

$$\alpha_m \int_t^{[t+k_m+1]} p(s) ds \leq \alpha_m \sum_{i=[t]}^{[t]+k_m} p_i \leq 1, \quad t \geq n_2. \quad (2.16)$$

Recall that $k_1 \leq k_2 \leq \cdots \leq k_m$, therefore

$$\alpha_j \int_t^{[t+k_j+1]} p(s) ds \leq \frac{\alpha_j}{\alpha_m}, \quad t \geq n_2, \quad j = 1, 2, \dots, m. \quad (2.17)$$

Substituting this into (2.13), we obtain

$$\sum_{j=1}^m \frac{\alpha_j}{\alpha_m} \int_{N-k_j}^N \lambda(t) dt \geq \int_{n_2}^N p(t) \ln A(t) dt, \quad (2.18)$$

or

$$\ln \left(\prod_{j=1}^m \left(\frac{x(N-k_j)}{x(N)} \right)^{\alpha_j} \right) \geq \alpha_m \int_{n_2}^N p(t) \ln A(t) dt. \quad (2.19)$$

It follows that

$$\lim_{N \rightarrow \infty} \prod_{i=1}^m \left(\frac{x(N-k_j)}{x(N)} \right)^{\alpha_j} \geq \exp \left(\alpha_m \int_{n_2}^{\infty} p(t) \ln A(t) dt \right). \quad (2.20)$$

Let $E = \{n \geq n_2 \mid p_n > 0\}$. Then

$$\begin{aligned} & \int_{n_2}^{\infty} p(t) \ln A(t) dt \\ &= \sum_{n=n_2}^{\infty} \int_n^{n+1} p(t) \ln \left(e \sum_{j=1}^m \alpha_j \int_t^{[t+k_j+1]} p(s) ds + 1 - \text{sign} \left(\sum_{j=1}^m \alpha_j \int_t^{[t+k_j+1]} p(s) ds \right) \right) dt \\ &= \sum_{n=n_2}^{\infty} p_n \int_n^{n+1} \ln \left(e \sum_{j=1}^m \alpha_j \left(\int_n^{n+k_j+1} p(s) ds - \int_n^t p(s) ds \right) + 1 \right. \\ & \quad \left. - \text{sign} \left(\sum_{j=1}^m \alpha_j \left(\int_n^{n+k_j+1} p(s) ds - \int_n^t p(s) ds \right) \right) \right) dt \\ &= \sum_{n=n_2}^{\infty} p_n \int_n^{n+1} \ln \left(e \sum_{j=1}^m \alpha_j \left(\sum_{i=n}^{n+k_j} p_i - p_n(t-n) \right) + 1 \right. \\ & \quad \left. - \text{sign} \left(\sum_{j=1}^m \alpha_j \left(\sum_{i=n}^{n+k_j} p_i - p_n(t-n) \right) \right) \right) dt \\ &= \sum_E p_n \int_n^{n+1} \ln \left(e \sum_{j=1}^m \alpha_j \left(\sum_{i=n}^{n+k_j} p_i - p_n(t-n) \right) \right) dt \end{aligned}$$

$$\begin{aligned}
&= \sum_E \left(\left(\sum_{j=1}^m \alpha_j \sum_{i=n}^{n+k_j} p_i \ln \left(\sum_{j=1}^m \alpha_j \sum_{i=n}^{n+k_j} p_i \right) \right) \right. \\
&\quad \left. - \sum_{j=1}^m \alpha_j \sum_{i=n+1}^{n+k_j} p_i \ln \left(\sum_{j=1}^m \alpha_j \sum_{i=n+1}^{n+k_j} p_i + 1 - \text{sign} \left(\sum_{j=1}^m \alpha_j \sum_{i=n+1}^{n+k_j} p_i \right) \right) \right) \\
&= \sum_{n=n_2}^{\infty} \left(\sum_{j=1}^m \alpha_j \sum_{i=n}^{n+k_j} p_i \ln \left(\sum_{j=1}^m \alpha_j \sum_{i=n}^{n+k_j} p_i + 1 - \text{sign} \left(\sum_{j=1}^m \alpha_j \sum_{i=n}^{n+k_j} p_i \right) \right) \right. \\
&\quad \left. - \sum_{j=1}^m \alpha_j \sum_{i=n+1}^{n+k_j} p_i \ln \left(\sum_{j=1}^m \alpha_j \sum_{i=n+1}^{n+k_j} p_i + 1 - \text{sign} \left(\sum_{j=1}^m \alpha_j \sum_{i=n+1}^{n+k_j} p_i \right) \right) \right).
\end{aligned} \tag{2.21}$$

From (2.1) and (2.20), we have

$$\lim_{N \rightarrow \infty} \prod_{j=1}^m \left(\frac{x(N-k_j)}{x(N)} \right)^{\alpha_j} = \infty. \tag{2.22}$$

On the other hand, it follows from (2.1) that

$$\limsup_{n \rightarrow \infty} p_n > 0. \tag{2.23}$$

By (2.15), we have

$$\limsup_{n \rightarrow \infty} p_n \leq \limsup_{n \rightarrow \infty} \frac{\mathcal{Y}_n}{\alpha_m \mathcal{Y}_{n-k_m}} = \frac{1}{\alpha_m} \frac{1}{\liminf_{n \rightarrow \infty} (\mathcal{Y}_{n-k_m} / \mathcal{Y}_n)}, \tag{2.24}$$

which, together with (2.23) yields

$$\liminf_{n \rightarrow \infty} \frac{\mathcal{Y}_{n-k_m}}{\mathcal{Y}_n} < \infty, \tag{2.25}$$

that is,

$$\liminf_{N \rightarrow \infty} \left(\frac{x(N-k_m)}{x(N)} \right)^{\alpha_m} < \infty, \tag{2.26}$$

and so

$$\liminf_{N \rightarrow \infty} \prod_{j=1}^m \left(\frac{x(N-k_j)}{x(N)} \right)^{\alpha_j} \leq \liminf_{N \rightarrow \infty} \frac{x(N-k_m)}{x(N)} < \infty, \tag{2.27}$$

which contradicts (2.22) and so the proof is completed. $\square$

From [Theorem 2.1](#) we have immediately.

COROLLARY 2.2. Assume that there exists an integer $N \geq 0$ such that

$$\sum_{j=1}^m \alpha_j \sum_{i=n}^{n+k_j} p_i > 0 \quad \text{for } n \geq N, \quad (2.28)$$

and that

$$\sum_{n=N}^{\infty} \left[\sum_{j=1}^m \alpha_j \sum_{j=n}^{n+k_j} p_i \ln \left(\sum_{j=1}^m \alpha_j \sum_{i=n}^{n+k_j} p_i \right) - \sum_{j=1}^m \alpha_j \sum_{i=n+1}^{n+k_j} p_i \ln \left(\sum_{j=1}^m \alpha_j \sum_{i=n+1}^{n+k_i} p_i \right) \right] = \infty. \quad (2.29)$$

Then every solution of (1.1) oscillates.

Clearly, when $k_1 = k_2 = \dots = k_m$, condition (2.1) reduces to (1.4).

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JIANCHU JIANG: DEPARTMENT OF MATHEMATICS, LOUDI TEACHERS' COLLEGE, LOUDI, HUNAN 417000, CHINA

E-mail address: ldjjc@mail.ld.hn.cn