ON n -FOLD FUZZY IMPLICATIVE/COMMUTATIVE IDEALS OF BCK-ALGEBRAS

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ABSTRACT. We consider the fuzzification of the notion of an n -fold implicative ideal, an n -fold (weak) commutative ideal. We give characterizations of an n -fold fuzzy implicative ideal. We establish an extension property for n -fold fuzzy commutative ideals.

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1. Introduction. Huang and Chen [1] introduced the notion of n -fold implicative ideals and n -fold (weak) commutative ideals. The aim of this paper is to discuss the fuzzification of n -fold implicative ideals, n -fold commutative ideals and n -fold weak commutative ideals. We show that every n -fold fuzzy implicative ideal is an n -fold fuzzy positive implicative ideal, and so a fuzzy ideal, and give a condition for a fuzzy ideal to be an n -fold fuzzy implicative ideal. Using the level set, we provide a characterization of an n -fold fuzzy implicative ideal. We also give a condition for a fuzzy ideal to be an n -fold fuzzy (weak) commutative ideal. We show that every n -fold fuzzy positive implicative ideal which is an n -fold fuzzy weak commutative ideal is an n -fold fuzzy implicative ideal. Finally, we establish an extension property for n -fold fuzzy commutative ideals.

2. Preliminaries. We include some elementary aspects of BCK-algebras that are necessary for this paper, and for more details we refer to [1, 2, 4, 5]. By a *BCK-algebra* we mean an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying the axioms:

- (I) $((x * y) * (x * z)) * (z * y) = 0$,
- (II) $(x * (x * y)) * y = 0$,
- (III) $x * x = 0$,
- (IV) $0 * x = 0$,
- (V) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y, z \in X$.

We can define a partial ordering $\leq$ on X by $x \leq y$ if and only if $x * y = 0$. In any BCK-algebra X , the following hold:

- (P1) $x * 0 = x$,
- (P2) $x * y \leq x$,
- (P3) $(x * y) * z = (x * z) * y$,
- (P4) $(x * z) * (y * z) \leq x * y$,
- (P5) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

Throughout, X will always mean a BCK-algebra unless otherwise specified. A non-empty subset I of X is called an *ideal* of X if it satisfies:

- (I1) $0 \in I$,

(I2) $x * y \in I$ and $y \in I$ imply $x \in I$.

A nonempty subset I of X is said to be an *implicative ideal* of X if it satisfies:

(I1) $0 \in I$,

(I3) $(x * (y * x)) * z \in I$ and $z \in I$ imply $x \in I$.

A nonempty subset I of X is said to be a *commutative ideal* of X if it satisfies:

(I1) $0 \in I$,

(I4) $(x * y) * z \in I$ and $z \in I$ imply $x * (y * (y * x)) \in I$.

We now review some fuzzy logic concepts. A fuzzy set in a set X is a function $\mu : X \rightarrow [0, 1]$. For a fuzzy set μ in X and $t \in [0, 1]$ define $U(\mu; t)$ to be the set $U(\mu; t) := \{x \in X \mid \mu(x) \geq t\}$.

A fuzzy set μ in X is said to be a *fuzzy ideal* of X if

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

Note that every fuzzy ideal μ of X is order reversing, that is, if $x \leq y$ then $\mu(x) \geq \mu(y)$.

A fuzzy set μ in X is called a *fuzzy implicative ideal* of X if it satisfies:

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F3) $\mu(x) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$ for all $x, y, z \in X$.

A fuzzy set μ in X is called a *fuzzy commutative ideal* of X if it satisfies:

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F4) $\mu(x * (y * (y * x))) \geq \min\{\mu((x * y) * z), \mu(z)\}$ for all $x, y, z \in X$.

3. n -fold fuzzy implicative ideals. For any elements x and y of a BCK-algebra X , $x * y^n$ denotes

$$(\cdots ((x * y) * y) * \cdots) * y \quad (3.1)$$

in which y occurs n times. Huang and Chen [1] introduced the concept of n -fold implicative ideals as follows.

DEFINITION 3.1 (see [1]). A subset A of X is called an *n -fold implicative ideal* of X if

(I1) $0 \in A$,

(I5) $(x * (y * x^n)) * z \in A$ and $z \in A$ imply $x \in A$ for every $x, y, z \in X$.

We consider the fuzzification of the concept of n -fold implicative ideal.

DEFINITION 3.2. A fuzzy set μ in X is called an *n -fold fuzzy implicative ideal* of X if

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F5) $\mu(x) \geq \min\{\mu((x * (y * x^n)) * z), \mu(z)\}$ for every $x, y, z \in X$.

Notice that the 1-fold fuzzy implicative ideal is a fuzzy implicative ideal.

THEOREM 3.3. Every n -fold fuzzy implicative ideal is a fuzzy ideal.

PROOF. The condition (F2) follows from taking $y = 0$ in (F5). □

The following example shows that the converse of Theorem 3.3 may not be true.

EXAMPLE 3.4. Let $X = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of natural numbers, in which the operation $*$ is defined by $x * y = \max\{0, x - y\}$ for all $x, y \in X$. Then X is a BCK-algebra (see [1, Example 1.3]). Let μ be a fuzzy set in X given by $\mu(0) = t_0 > t_1 = \mu(x)$ for all $x (\neq 0) \in X$. Then μ is a fuzzy ideal of X . But μ is not a 2-fold fuzzy implicative ideal of X because

$$\mu(3) = t_1 < t_0 = \mu(0) = \min\{\mu((3 * (14 * 3^2)) * 0), \mu(0)\}. \quad (3.2)$$

We give a condition for a fuzzy ideal to be an n -fold fuzzy implicative ideal.

THEOREM 3.5. A fuzzy ideal μ of X is n -fold fuzzy implicative if and only if $\mu(x) \geq \mu(x * (y * x^n))$ for all $x, y \in X$.

PROOF. Necessity is by taking $z = 0$ in (F5). Suppose that a fuzzy ideal μ satisfies the inequality $\mu(x) \geq \mu(x * (y * x^n))$ for all $x, y \in X$. Then

$$\mu(x) \geq \mu(x * (y * x^n)) \geq \min\{\mu((x * (y * x^n)) * z), \mu(z)\}. \quad (3.3)$$

Hence μ is an n -fold fuzzy implicative ideal of X . $\square$

THEOREM 3.6. A fuzzy set μ in X is an n -fold fuzzy implicative ideal of X if and only if the nonempty level set $U(\mu; t)$ of μ is an n -fold implicative ideal of X for every $t \in [0, 1]$.

PROOF. Assume that μ is an n -fold fuzzy implicative ideal of X and $U(\mu; t) \neq \emptyset$ for every $t \in [0, 1]$. Then there exists $x \in U(\mu; t)$. It follows from (F1) that $\mu(0) \geq \mu(x) \geq t$ so that $0 \in U(\mu; t)$. Let $x, y, z \in X$ be such that $(x * (y * x^n)) * z \in U(\mu; t)$ and $z \in U(\mu; t)$. Then $\mu((x * (y * x^n)) * z) \geq t$ and $\mu(z) \geq t$, which imply from (F5) that

$$\mu(x) \geq \min\{\mu((x * (y * x^n)) * z), \mu(z)\} \geq t \quad (3.4)$$

so that $x \in U(\mu; t)$. Therefore $U(\mu; t)$ is an n -fold implicative ideal of X . Conversely, suppose that $U(\mu; t) (\neq \emptyset)$ is an n -fold implicative ideal of X for every $t \in [0, 1]$. For any $x \in X$, let $\mu(x) = t$. Then $x \in U(\mu; t)$. Since $0 \in U(\mu; t)$, we get $\mu(0) \geq t = \mu(x)$ and so $\mu(0) \geq \mu(x)$ for all $x \in X$. Now assume that there exist $a, b, c \in X$ such that

$$\mu(a) < \min\{\mu((a * (b * a^n)) * c), \mu(c)\}. \quad (3.5)$$

Selecting $s_0 = (1/2)(\mu(a) + \min\{\mu((a * (b * a^n)) * c), \mu(c)\})$, then

$$\mu(a) < s_0 < \min\{\mu((a * (b * a^n)) * c), \mu(c)\}. \quad (3.6)$$

It follows that $(a * (b * a^n)) * c \in U(\mu; s_0)$, $c \in U(\mu; s_0)$, and $a \notin U(\mu; s_0)$. This is a contradiction. Hence μ is an n -fold fuzzy implicative ideal of X . $\square$

DEFINITION 3.7 (see [3]). A fuzzy set μ in X is called an n -fold fuzzy positive implicative ideal of X if

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F6) $\mu(x * y^n) \geq \min\{\mu((x * y^{n+1}) * z), \mu(z)\}$ for all $x, y, z \in X$.

LEMMA 3.8 (see [3, Theorem 3.13]). *Let μ be a fuzzy set in X . Then μ is an n -fold fuzzy positive implicative ideal of X if and only if the nonempty level set $U(\mu; t)$ of μ is an n -fold positive implicative ideal of X for every $t \in [0, 1]$.*

LEMMA 3.9 (see [1, Theorem 2.5]). *Every n -fold implicative ideal is an n -fold positive implicative ideal.*

Using Theorem 3.6 and Lemmas 3.8 and 3.9, we have the following theorem.

THEOREM 3.10. *Every n -fold fuzzy implicative ideal is an n -fold fuzzy positive implicative ideal.*

4. n -fold fuzzy commutative ideals

DEFINITION 4.1 (see [1]). A subset A of X is called an n -fold commutative ideal of X if

- (I1) $0 \in A$,
- (I6) $(x * y) * z \in A$ and $z \in A$ imply $x * (y * (y * x^n)) \in A$ for all $x, y, z \in X$.

A subset A of X is called an n -fold weak commutative ideal of X if

- (I1) $0 \in A$,
- (I7) $(x * (x * y^n)) * z \in A$ and $z \in A$ imply $y * (y * x) \in A$ for all $x, y, z \in X$.

We consider the fuzzification of n -fold (weak) commutative ideals as follows.

DEFINITION 4.2. A fuzzy set μ in X is called an n -fold fuzzy commutative ideal of X if

- (F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,
- (F7) $\mu(x * (y * (y * x^n))) \geq \min\{\mu((x * y) * z), \mu(z)\}$ for all $x, y, z \in X$.

A fuzzy set μ in X is called an n -fold fuzzy weak commutative ideal of X if

- (F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,
- (F8) $\mu(y * (y * x)) \geq \min\{\mu((x * (x * y^n)) * z), \mu(z)\}$ for all $x, y, z \in X$.

Note that the 1-fold fuzzy commutative ideal is a fuzzy commutative ideal. Putting $y = 0$ and $y = x$ in (F7) and (F8), respectively, we know that every n -fold fuzzy commutative (or fuzzy weak commutative) ideal is a fuzzy ideal.

THEOREM 4.3. *Let μ be a fuzzy ideal of X . Then*

- (i) μ is an n -fold fuzzy commutative ideal of X if and only if

$$\mu(x * (y * (y * x^n))) \geq \mu(x * y) \quad \forall x, y \in X. \quad (4.1)$$

- (ii) μ is an n -fold fuzzy weak commutative ideal of X if and only if

$$\mu(y * (y * x)) \geq \mu(x * (x * y^n)) \quad \forall x, y \in X. \quad (4.2)$$

PROOF. The proof is straightforward. □

LEMMA 4.4 (see [3, Theorem 3.12]). *A fuzzy set μ in X is an n -fold fuzzy positive implicative ideal of X if and only if μ is a fuzzy ideal of X in which the following inequality holds:*

$$(F9) \quad \mu((x * z^n) * (y * z^n)) \geq \mu((x * y) * z^n) \quad \forall x, y, z \in X.$$

THEOREM 4.5. *If μ is both an n -fold fuzzy positive implicative ideal and an n -fold fuzzy weak commutative ideal of X , then it is an n -fold fuzzy implicative ideal of X .*

PROOF. Let $x, y \in X$. Using [Theorem 4.3\(ii\)](#), [Lemma 4.4](#), (P3), and (III), we have

$$\begin{aligned} \mu(x * (x * (y * x^n))) &\geq \mu((y * x^n) * ((y * x^n) * x^n)) \\ &\geq \mu((y * (y * x^n)) * x^n) \\ &= \mu((y * x^n) * (y * x^n)) \\ &= \mu(0). \end{aligned} \tag{4.3}$$

It follows from (F1) and (F2) that

$$\begin{aligned} \mu(x) &\geq \min\{\mu(x * (x * (y * x^n))), \mu(x * (y * x^n))\} \\ &\geq \min\{\mu(0), \mu(x * (y * x^n))\} \\ &= \mu(x * (y * x^n)) \end{aligned} \tag{4.4}$$

so from [Theorem 3.5](#), μ is an n -fold fuzzy implicative ideal of X . $\square$

THEOREM 4.6 (extension property for n -fold fuzzy commutative ideals). *Let μ and ν be fuzzy ideals of X such that $\mu(0) = \nu(0)$ and $\mu \subseteq \nu$, that is, $\mu(x) \leq \nu(x)$ for all $x \in X$. If μ is an n -fold fuzzy commutative ideal of X , then so is ν .*

PROOF. Let $x, y \in X$. Taking $u = x * (x * y)$, we have

$$\begin{aligned} \nu(0) &= \mu(0) = \mu(u * y) \\ &\leq \mu(u * (y * (y * u^n))) \\ &\leq \nu(u * (y * (y * u^n))) \\ &= \nu((x * (x * y)) * (y * (y * u^n))) \\ &= \nu((x * (y * (y * u^n))) * (x * y)). \end{aligned} \tag{4.5}$$

Since $x * (y * (y * x^n)) \leq x * (y * (y * u^n))$ and since ν is order reversing, it follows that

$$\begin{aligned} \nu(x * (y * (y * x^n))) &\geq \nu(x * (y * (y * u^n))) \\ &\geq \min\{\nu((x * (y * (y * u^n))) * (x * y)), \nu(x * y)\} \\ &\geq \min\{\nu(0), \nu(x * y)\} \\ &= \nu(x * y). \end{aligned} \tag{4.6}$$

Hence, by [Theorem 4.3\(i\)](#), ν is an n -fold fuzzy commutative ideal of X . $\square$

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