PRODUCTS OF PROTOPOLOGICAL GROUPS

JULIE C. JONES

(Received 30 March 2001)

ABSTRACT. Montgomery and Zippin said that a group is approximated by Lie groups if every neighborhood of the identity contains an invariant subgroup H such that G/H is topologically isomorphic to a Lie group. Bagley, Wu, and Yang gave a similar definition, which they called a pro-Lie group. Covington extended this concept to a protopological group. Covington showed that protopological groups possess many of the characteristics of topological groups. In particular, Covington showed that in a special case, the product of protopological groups is a protopological group. In this note, we give a characterization theorem for protopological groups and use it to generalize her result about products to the category of all protopological groups.

2000 Mathematics Subject Classification. 22A99.

1. Introduction. Montgomery and Zippin [5] said that a group is *approximated by Lie groups* if every neighborhood of the identity contains an invariant subgroup H such that G/H is topologically isomorphic to a Lie group. Using a similar idea, Bagley, Wu, and Yang [1] defined a pro-Lie group. Covington [3] extended this concept to topological groups. She defined a protopological group as a group G with a topology τ and a collection $\mathcal{N}$ of normal subgroups such that (1) for every neighborhood U of the identity, there exists $N \in \mathcal{N}$ such that $N \subseteq U$ and (2) G/N with the quotient topology is a topological group for every $N \in \mathcal{N}$. The collection $\mathcal{N}$ is called a normal system. We denote the quotient topology on G/N by $q_N(\tau)$, and we call the collection $\mathfrak{Q} = \{q_N(\tau)\}_{N \in \mathcal{N}}$ a quotient system for (G, τ) . In [2], Covington defines a t -protopological group as a protopological group (G, τ) with the additional requirement that the natural map $\eta_N : G \rightarrow G/N$ is an open map for all $N \in \mathcal{N}$. She also shows that the product of t -protopological groups is a t -protopological group. Although her proof uses ideas different than those used in the proof that a product of topological groups is a topological group, it uses the fact that $\eta_N : G \rightarrow G/N$ is an open map for all $N \in \mathcal{N}$. Since there are protopological groups, which are not t -protopological, it is of interest to determine if the category of protopological groups is closed under products. However, it is apparent that a different proof technique is needed, since we do not have the hypothesis that $\eta_N : G \rightarrow G/N$ is an open map for all $N \in \mathcal{N}$. In this note, we will give a characterization theorem for protopological groups, and then use it to show that the product of protopological groups is protopological.

Let (G, τ) be a protopological group with normal system $\mathcal{N}$ and quotient system $\mathfrak{Q} = \{q_N(\tau)\}_{N \in \mathcal{N}}$. We note that for each $N \in \mathcal{N}$, the pullback topology from $(G/N, q_N(\tau))$ determines a group topology on G . We will denote this topology on G by $P_N(\tau)$. Since the join of group topologies is a group topology, $\tau_p = \vee_{N \in \mathcal{N}} P_N(\tau)$ is also a group

topology on G . We call τ_p the complete pullback topology on G generated by τ . This topology may also be called the weak topology on G . Covington [2] showed that τ_p is the Graev topology when (G, τ) is a prototopological group. It is a well-known result, due to Hewitt and Ross [4], that $\tau_p = \bigvee_{N \in \mathcal{N}} P_N(\tau)$ is the coarsest topology that makes G a topological group. Hence, for a prototopological group (G, τ) with normal system $\mathcal{N}$, the complete pullback topology τ_p is the only group topology contained in τ . When using the pullback topologies, we will be interested in saturated sets. In particular, we will say that a set $U \subseteq G$ is *saturated with respect to* $N \in \mathcal{N}$ if for all $x \in U$, $\eta_N^{-1}(\eta_N(x)) \subseteq U$.

2. Characterization and product theorems

THEOREM 2.1 (characterization theorem for prototopological groups). *Let (G, τ) be a prototopological group with normal system $\mathcal{N}$ and quotient system $\mathcal{Q} = \{q_N(\tau)\}_{N \in \mathcal{N}}$. Let τ^* be a topology on G . Then (G, τ^*) is a prototopological group with normal system $\mathcal{N}$ and quotient system $\mathcal{Q}$ if and only if τ^* satisfies the following properties:*

- (a) $\tau_p \subseteq \tau^*$;
- (b) if $U \in \tau^*$, then $U \in \tau_p$ or U is not saturated with respect to N for all $N \in \mathcal{N}$; and
- (c) if $U \in \tau^*$ is a neighborhood of e , then there exists $N \in \mathcal{N}$ such that $N \subseteq U$.

PROOF. Let (G, τ^*) be a prototopological group. Since $\tau_p = \bigvee_{N \in \mathcal{N}} P_N(\tau) = \bigvee_{N \in \mathcal{N}} P_N(\tau^*)$ is the coarsest topology that makes G a topological group [4], it follows that $\tau_p \subseteq \tau^*$. If $U \in \tau^*$ is saturated with respect to some $N \in \mathcal{N}$, then $U = \eta_N^{-1}(\eta_N(U))$. But then $U \in P_N(\tau) \subseteq \tau_p$. Now, if $U \in \tau^*$ is a neighborhood of e , there exists $N \in \mathcal{N}$ such that $N \subseteq U$. Conversely, assume that (a), (b), and (c) are satisfied. For $N \in \mathcal{N}$, consider the group G/N with the topology $q_N(\tau^*)$. Let $U \in q_N(\tau) = q_N(\tau_p)$. Since $\eta_N^{-1}(U) \in \tau_p \subseteq \tau^*$, it follows that $U \in q_N(\tau^*)$. Hence, $q_N(\tau) = q_N(\tau_p) \subseteq q_N(\tau^*)$. Now, let $U \in q_N(\tau^*)$. Then $q_N^{-1}(U) \in \tau_p$ and $q_N^{-1}(U)$ is saturated with respect to N . Therefore, $q_N^{-1}(U) \in \tau_p$ which implies that $U \in q_N(\tau_p) = q_N(\tau)$. Thus, $q_N(\tau^*) \subseteq q_N(\tau_p) = q_N(\tau)$. Therefore, $q_N(\tau^*) = q_N(\tau)$ for all $N \in \mathcal{N}$. $\square$

By imposing the additional condition that $\eta_N(\tau^*) \subseteq q_N(\tau)$, we obtain a characterization theorem for t -prototopological groups.

THEOREM 2.2 (product theorem). *Let (G_α, τ_α) be a prototopological group with normal system $\mathcal{N}_\alpha$ and quotient system $\mathcal{Q}_\alpha = \{q_N(\tau_\alpha)\}_{N \in \mathcal{N}_\alpha}$, for all $\alpha \in A$. Let $G = \prod_{\alpha \in A} G_\alpha$, and let $\tau = \prod_{\alpha \in A} \tau_\alpha$ be the product topology on G . Then (G, τ) is a prototopological group with normal system $\mathcal{N} = \{\prod_{\alpha \in A} N_\alpha \mid N_\alpha \in \mathcal{N}_\alpha \text{ for all } \alpha \in A \text{ and } N_\alpha = G_\alpha \text{ for all but finitely many } \alpha \in A\}$.*

PROOF. For each $\alpha \in A$, let τ_{p_α} be the complete pullback topology on G_α . Then $\{(G_\alpha, \tau_{p_\alpha})\}_{\alpha \in A}$ is a collection of topological groups, and (G, τ_p) is a topological group, where $\tau_p = \prod_{\alpha \in A} \tau_{p_\alpha}$ is the product topology of $\{\tau_{p_\alpha}\}_{\alpha \in A}$. By the characterization theorem, $\tau_{p_\alpha} \subseteq \tau_\alpha$ for each $\alpha \in A$. So, $\tau_p = \prod_{\alpha \in A} \tau_{p_\alpha} \subseteq \prod_{\alpha \in A} \tau_\alpha = \tau$. Now, if $U \in \tau$ is a neighborhood of $e = \langle e_\alpha \rangle_{\alpha \in A} \in G$ then there exists $U_{\alpha_i} \in \tau_{\alpha_i}$, for $i = 1, \dots, n$, such that $e \in \prod_{i=1}^n U_{\alpha_i} \times \prod_{\alpha \notin \{\alpha_1, \dots, \alpha_n\}} G_\alpha \subseteq U$. Then for each $\alpha_i \in \{\alpha_1, \dots, \alpha_n\}$, there exists $N_{\alpha_i} \in \mathcal{N}_{\alpha_i}$ with $N_{\alpha_i} \subseteq U_{\alpha_i}$. Hence, $N = \prod_{i=1}^n N_{\alpha_i} \times \prod_{\alpha \notin \{\alpha_1, \dots, \alpha_n\}} G_\alpha \in \mathcal{N}$ and $e \in N \subseteq \prod_{i=1}^n U_{\alpha_i} \times \prod_{\alpha \notin \{\alpha_1, \dots, \alpha_n\}} G_\alpha \subseteq U$. Since (G, τ_p) is a topological group, $(G/N, q_N(\tau_p))$ is a

topological group for all $N \in \mathcal{N}$. Hence, (G, τ_p) is a prototopological group with normal system $\mathcal{N}$ and quotient system $\mathcal{Q} = \{q_N(\tau_p)\}_{N \in \mathcal{N}}$. For each $N \in \mathcal{N}$, let $P_N(\tau_p)$ be the pullback topology on G from $(G/N, q_N(\tau_p))$. Since the complete pullback topology is the only group topology that makes G a prototopological group with normal system $\mathcal{N}$ and quotient system $\mathcal{Q}$, we have that $\tau_p = \bigvee_{N \in \mathcal{N}} P_N(\tau_p)$. For $N \in \mathcal{N}$, let $\eta_N : G \rightarrow G/N$ be defined by $\eta_N(g) = gN$. Now, if $U \in \tau$ is saturated with respect to some $N \in \mathcal{N}$, then, $\eta_N(U) \subseteq q_N(\tau_p)$. But then, $U = \eta_N^{-1}(\eta_N(U)) \in P_N(\tau_p) \subseteq \tau_p$. So, $U \in \tau_p$. By the characterization theorem, (G, τ) is a prototopological group. $\square$

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JULIE C. JONES: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LOUISIANA AT LAFAYETTE, LAFAYETTE, LA 70504, USA

E-mail address: jcj6904@louisiana.edu