ROUGH MARCINKIEWICZ INTEGRAL OPERATORS

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Abstract. We study the Marcinkiewicz integral operator
\[ M_\Phi f(x) = \left( \int_\mathbb{R}^n \left( \int_{|y| \leq 2t} |f(x - \Phi(y))\Omega(y)/|y|^{n-1}|dy \right)^2 dt / 2t \right)^{1/2}, \]
where \( \Phi \) is a polynomial mapping from \( \mathbb{R}^n \) into \( \mathbb{R}^d \) and \( \Omega \) is a homogeneous function of degree zero on \( \mathbb{R}^n \) with mean value zero over the unit sphere \( S^{n-1} \). We prove an \( L^p \) boundedness result of \( M_\Phi \) for rough \( \Omega \).

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1. Introduction. Let \( \mathbb{R}^n \), \( n \geq 2 \) be the \( n \)-dimensional Euclidean space and \( S^{n-1} \) be the unit sphere in \( \mathbb{R}^n \) equipped with the induced Lebesgue measure. Consider the Marcinkiewics integral operator
\[ \mu f(x) = \left( \int_{-\infty}^{\infty} |F_t(x)|^2 dt / 2t \right)^{1/2}, \]
where
\[ F_t(x) = \int_{|x-y| \leq 2t} f(y) \Omega(x-y) / |x-y|^{n-1} dy, \]
and \( \Omega \) is a homogeneous function of degree zero which has the following properties:
\[ \Omega \in L^1(S^{n-1}), \quad \int_{S^{n-1}} \Omega(y') d\sigma(y') = 0. \]

When \( \Omega \in \text{Lip}_\alpha(S^{n-1}), \) \( 0 < \alpha < 1 \), Stein proved the \( L^p \) boundedness of \( \mu(f) \) for all \( 1 < p \leq 2 \). Subsequently, Benedek, Calderón, and Panzone proved the \( L^p \) boundedness of \( \mu(f) \) for all \( 1 < p < \infty \) under the condition \( \Omega \in C(S^{n-1}) \) (see [2]).

The authors of [3] were able to prove the following result for the more general class of operators
\[ \mu_P f(x) = \left( \int_{-\infty}^{\infty} |F_{P,t}(x)|^2 dt / 2t \right)^{1/2}, \]
where
\[ F_{P,t}(x) = \int_{|y| \leq 2t} f(x - P(|y|)y') \Omega(y) / |y|^{n-1} dy \]
and \( P \) is a real-valued polynomial on \( \mathbb{R} \) and satisfies \( P(0) = 0 \).

**Theorem 1.1** (see [3]). Let \( \alpha > 0 \), and \( \Omega \in V_\alpha(n) \). Then the operator \( \mu_P \) is bounded in \( L^p(\mathbb{R}^n) \) for \( (2\alpha + 2) / (2\alpha + 1) < p < 2 + 2\alpha \).
In [1], Al-Salman and Pan studied the singular integral operator

$$T_{\Omega,\mathcal{P}} f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - \mathcal{P}(y)) \frac{\Omega(y')}{|y'|^n} \, dy,$$  \hspace{1cm} (1.6)

where $\mathcal{P} = (P_1, \ldots, P_d): \mathbb{R}^n \to \mathbb{R}^d$ is a polynomial mapping, $d \geq 1$, $n \geq 2$. The authors of [1] proved that $T_{\Omega,\mathcal{P}}$ is bounded in $L^p(\mathbb{R}^d)$ whenever $(2 + 2\alpha)/(1 + 2\alpha) < p < 2 + 2\alpha$ and $\Omega \in W_\alpha(n)$. Here $W_\alpha(n)$ is a subspace of $L^1(S^{n-1})$ and its definition as well as the definition of $V_\alpha(n)$ will be reviewed in Section 2. It was shown in [1] that $W_\alpha(n) = V_\alpha(n)$ if $n = 2$ and it is a proper subspace of $V_\alpha(n)$ if $n \geq 3$.

Our purpose in this paper is to study the $L^p$ boundedness of the class of operators

$$M_{\mathcal{P}} f(x) = \left( \int_{-\infty}^{\infty} |F_{\mathcal{P},t}(x)|^2 \frac{dt}{2\pi} \right)^{1/2},$$  \hspace{1cm} (1.7)

where

$$F_{\mathcal{P},t}(x) = \int_{|y| \leq 2^t} f(x - \mathcal{P}(y)) \frac{\Omega(y)}{|y|^{n-1}} \, dy.$$  \hspace{1cm} (1.8)

Our main result in this paper is the following theorem.

**Theorem 1.2.** Let $\alpha > 0$, and $\Omega \in W_\alpha(n)$. Then the operator $M_{\mathcal{P}}$ is bounded in $L^p(\mathbb{R}^d)$ for $(2\alpha + 2)/(2\alpha + 1) < p < 2 + 2\alpha$. The bound of $M_{\mathcal{P}} f$ is independent of the coefficients of $\{P_j\}$.

By [1, Theorem 3.1] and Theorem 1.2 we have the following corollary.

**Corollary 1.3.** Let $\alpha > 0$, $\Omega \in V_\alpha(2)$ and $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^d$. Then $M_{\mathcal{P}}$ is bounded in $L^p(\mathbb{R}^d)$ for $(2\alpha + 2)/(2\alpha + 1) < p < 2 + 2\alpha$. The bound of $M_{\mathcal{P}}$ is independent of the coefficients of $\{P_j\}$.

2. Preparation. We start this section by recalling the following definition from [1].

**Definition 2.1.** For $\alpha > 0$, $N \geq 1$, let $\tilde{\mathcal{V}}(n,N) = \bigcup_{m=1}^{N} \mathcal{V}(n,m)$ and let $W_\alpha(N,n)$ be the subspace of $L^1(S^{n-1})$ defined by

$$W_\alpha(N,n) = \left\{ \Omega \in L^1(S^{n-1}) : \int_{S^{n-1}} \Omega(y') \, d\sigma(y') = 0, \, M_\alpha(N,n) < \infty \right\},$$  \hspace{1cm} (2.1)

where

$$M_\alpha(N,n) = \max \left\{ \int_{S^{n-1}} |\Omega(y')| \left( \log \frac{1}{|P(y')|} \right)^{1+\alpha} \, d\sigma(y') : P \in \tilde{\mathcal{V}}(n,N) \text{ with } \|P\| = 1 \right\}.\hspace{1cm} (2.2)$$

For $\alpha > 0$, we define $W_\alpha(n)$ to be

$$W_\alpha(n) = \bigcap_{N=1}^{\infty} W_\alpha(N,n).\hspace{1cm} (2.3)$$

Also, for $\alpha > 0$, we define $V_\alpha(n)$ by $V_\alpha(n) = W_\alpha(1,n)$ (see [6]).
Here \( \mathcal{Y}(n,m) \) is the space of all real-valued homogeneous polynomials on \( \mathbb{R}^n \) with degree equal to \( m \) and with norm \( \| \cdot \| \) defined by

\[
\left\| \sum_{|\alpha|=m} a_\alpha y^\alpha \right\| = \sum_{|\alpha|=m} |a_\alpha|.
\]

(2.4)

Now we need to recall the following results.

**Lemma 2.2** (see van der Corput [7]). Suppose \( \varphi \) and \( \psi \) are real-valued and smooth in \( (a,b) \), and that \( |\varphi^{(k)}(t)| \geq 1 \) for all \( t \in (a,b) \). Then the inequality

\[
\left| \int_a^b e^{-i\lambda \varphi(t)} \psi(t) \, dt \right| \leq C_k |\lambda|^{-1/k} \left[ |\psi(b)| + \int_a^b |\psi'(t)| \, dt \right],
\]

(2.5)

holds when

(i) \( k \geq 2 \), or

(ii) \( k = 1 \) and \( \varphi' \) is monotonic.

The bound \( C_k \) is independent of \( a, b, \varphi, \) and \( \lambda \).

**Lemma 2.3** (see [7]). Let \( \mathcal{P} = (P_1, \ldots, P_d) \) be a polynomial mapping from \( \mathbb{R}^n \) into \( \mathbb{R}^d \). Let \( \deg(\mathcal{P}) = \max_{1 \leq j \leq d} \deg(P_j) \). Suppose \( \Omega \in L^1(S^{n-1}) \) and

\[
\mu_{\Omega,\mathcal{P}}(x) = \sup_{h>0} \left| \frac{1}{h^n} \int_{|y|<h} f(x - \mathcal{P}(y)) \Omega(y') \, dy \right|.
\]

(2.6)

Then for every \( 1 < p \leq \infty \), there exists a constant \( C_p > 0 \) which is independent of \( \Omega \) and the coefficients of \( \{P_j\} \) such that

\[
\left\| \mu_{\Omega,\mathcal{P}} f \right\|_p \leq C_p \|\Omega\|_{L^1(S^{n-1})} \|f\|_p
\]

(2.7)

for every \( f \in L^p(\mathbb{R}^d) \).

To each polynomial mapping \( \mathcal{P} = (P_1, \ldots, P_d) : \mathbb{R}^n \to \mathbb{R}^d \) with

\[
\deg(\mathcal{P}) = \max_{1 \leq j \leq d} \deg(P_j) = N, \quad d \geq 1, \quad n \geq 2,
\]

we define a family of measures

\[
\{ \varrho^l_t, \lambda^l_t : l = 0, 1, \ldots, N, \ t \in \mathbb{R} \}
\]

(2.9)

as follows.

For \( 1 \leq j \leq d, 0 \leq l \leq N \) let \( P_j = \sum_{|\alpha| \leq N} C_{j\alpha} y^\alpha \) and let \( Q^l_j = (Q^l_1, \ldots, Q^l_d) \) where \( Q^l_j = \sum_{|\alpha| \leq l} C_{j\alpha} y^\alpha \).

Now for \( 0 \leq l \leq N \) and \( t \in \mathbb{R} \), let \( \varrho^l_t \) and \( \lambda^l_t \) be the measures defined in the Fourier transform side by

\[
(\varrho^l_t)(\xi) = \int_{|y| \leq 2^l} e^{-2\pi i x \cdot Q^l(y)} \Omega(y') \, dy
\]

\[
(\lambda^l_t)(\xi) = \int_{|y| \leq 2^l} e^{-2\pi i x \cdot Q^l(y)} \left| \Omega(y') \right| \, dy.
\]

(2.10)
The maximal functions $(\vartheta_l)^*$ defined by
\[
(\vartheta_l)^*(f)(x) = \sup_{t \in \mathbb{R}} |\lambda_t^l * f(x)|,
\] (2.11)
for $l = 0, 1, \ldots, N$.

For later purposes, we need the following definition.

**Definition 2.4.** For each $1 \leq l \leq N$, let $N_l = \{ \alpha \in \mathbb{N}^n : |\alpha| = l \}$ and let $\{ \alpha \in \mathbb{N}^n : |\alpha| = l \} = \{ \alpha_1, \ldots, \alpha_{N_l} \}$. For each $1 \leq l \leq N$, define the linear transformations $L_{\alpha_j}^l : \mathbb{R}^d \to \mathbb{R}$ and $L_l : \mathbb{R}^d \to \mathbb{R}^{N_l}$ by
\[
L_{\alpha_j}^l (\xi) = \sum_{i=1}^{d} (C_{i,\alpha_j} \lambda_{\alpha_j}) \xi_i, \quad j = 1, \ldots, N_l,
\]
\[
L_l (\xi) = (L_{\alpha_1}^l (\xi), \ldots, L_{\alpha_{N_l}}^l (\xi)).
\] (2.12)

To simplify the proof of our result we need the following lemma.

**Lemma 2.5.** Let $\{ \sigma_l^t : l = 0, 1, \ldots, N, t \in \mathbb{R} \}$ be a family of measures such that $\sigma_0^t = 0$ for all $t \in \mathbb{R}$. Let $D_l : \mathbb{R}^n \to \mathbb{R}^d$, $l = 0, 1, \ldots, N$ be linear transformations. Suppose that for all $t \in \mathbb{R}$ and $l = 0, 1, \ldots, N$, then
\[
||\sigma_t^l|| \leq C(l),
\]
\[
|\hat{\sigma}_t^l| \leq C \frac{M_\alpha}{(\log [c2^{lt} |D_l(\xi)|])^{1+\alpha}},
\] (2.13)
\[
|\hat{\sigma}_t^l - \hat{\sigma}_{t-1}^l| \leq C2^{lt} |D_l(\xi)|.
\]

Then there exists a family of measures $\{ \nu_t^l : l = 1, \ldots, N \}_{t \in \mathbb{R}}$ such that
\[
||\nu_t^l|| \leq C(l),
\]
\[
|\hat{\nu}_t^l| \leq C \frac{M_\alpha}{(\log [c2^{lt} |D_l(\xi)|])^{1+\alpha}},
\] (2.14)
\[
|\hat{\nu}_t^l| \leq C2^{lt} |D_l(\xi)|,
\]
\[
\sigma_t^N = \sum_{l=1}^{N} \nu_t^l.
\]

**Proof.** By [5, Lemma 6.1], for each $l = 1, \ldots, N$ choose two nonsingular linear transformations
\[
A_l : \mathbb{R}^{r(l)} \to \mathbb{R}^d, \quad B_l : \mathbb{R}^d \to \mathbb{R}^d,
\] (2.15)
such that
\[
|A_l \pi_{r(l)}^d B_l(\xi)| \leq |D_l(\xi)| \leq N |A_l \pi_{r(l)}^d B_l(\xi)|, \quad \xi \in \mathbb{R}^d,
\] (2.16)
where $r(l) = \text{rank}(D_l)$ and $\pi_{r(l)}^d$ is the projection operator from $\mathbb{R}^d$ into $\mathbb{R}^{r(l)}$. 
Now choose $\eta \in C_0^\infty(\mathbb{R})$ such that $\eta(t) = 1$ for $|t| \leq 1/2$ and $\eta(t) = 0$ for $|t| \geq 1$. Let $\varphi(t) = \varphi(t^2)$ and let

$$
\nu_l^j(\xi) = \sigma_l^j(\xi) \prod_{l < j \leq N} \varphi(|2^{l}A_j \pi_{r(j)}^d B_j(\xi)|)
$$

$$
- (\sigma_l^{j-1}(\xi) \prod_{l-1 < j \leq N} \varphi(|2^{l}A_j \pi_{r(j)}^d B_j(\xi)|))
$$

with the convention $\prod_{j \in \emptyset} a_j = 1$, $1 \leq l \leq N$.

Hence, one can easily see that $\{\sigma_l^j : l = 1, \ldots, N, \ t \in \mathbb{R}\}$ is the desired family of measures.

Now for the boundedness of the maximal functions $(\vartheta_l)^*, l = 0, 1, \ldots, N$, we have the following lemma whose proof is an easy consequence of Lemma 2.3, polar coordinates and Hölder’s inequality:

**Lemma 2.6.** For $l = 1, \ldots, N$ and $p \in (1, \infty)$, there exists a constant $C_{p,l}$ which is independent of the coefficients of the polynomial components of the mapping $Q_l$ such that

$$
\|\vartheta_l^* f\|_p \leq C_{p,l} \|f\|_p.
$$

**3. Boundedness of some square functions.** For a nonnegative $C^\infty$ radial function $\Phi$ on $\mathbb{R}^n$ with

$$
\text{supp}(\Phi) \subset \{x \in \mathbb{R}^n : \frac{1}{2} \leq |x| \leq 2\}, \quad \int_0^\infty \frac{\Phi(t)}{t} dt = 1,
$$

and for a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^d$, define the functions $\psi_t, t \in \mathbb{R}$ by $\hat{\psi}_t(y) = \Phi(2^t L(y))$.

For a family of measures $\{\sigma_t\}_{t \in \mathbb{R}}$, real number $u$ and $l \in \mathbb{N}$, let $J_l^u(f)$ be the square function defined by

$$
J_l^u(f)(x) = \left( \int_{-\infty}^{\infty} |\sigma_t * \psi_{t+u} * f(x)|^2 dt \right)^{1/2}.
$$

For such a square function we have the following theorem.

**Theorem 3.1.** If $\{\sigma_t\}_{t \in \mathbb{R}}$ is a family of measures such that the corresponding maximal function

$$
\sigma^*(f)(x) = \sup_{t \in \mathbb{R}} \|\sigma_t * f(x)\|
$$

is bounded on $L^p(\mathbb{R}^d)$ for every $1 < p < \infty$, then

$$
\|J_l^u(f)\|_{L^p(\mathbb{R}^d)} \leq C_{p,l} \|\sigma^*\|_{(p/2)'} \sup_{t \in \mathbb{R}} \|\sigma_t\| \|f\|_{L^p(\mathbb{R}^d)}
$$

for every $1 < p < \infty$. Here $C_{p,l}$ is a constant that depends only on $p$ and the dimension of the underlying space.
Proof. If $\sup_{t \in \mathbb{R}} \| \sigma_t \| = \infty$, then the inequality holds trivially. Thus we may assume that $\sup_{t \in \mathbb{R}} \| \sigma_t \| < \infty$. In this case we follow a similar argument as in [4]. Let $p > 2$ and $q = \left( \frac{p}{2} \right)'$. Choose a nonnegative function $v \in L^q$ with $\| v \|_q = 1$ such that

$$\| J^l_u(f) \|_p^2 = \int_{\mathbb{R}^d} \left( \int_{-\infty}^{\infty} | \sigma_t * \psi_{l(t+u)} * f(x) |^2 dt \right) v(x) dx. \quad (3.5)$$

Thus it is easy to see that

$$\| J^l_u(f) \|_p^2 \leq \sup_{t \in \mathbb{R}} \| \sigma_t \| \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} | \sigma_t * \psi_{l(t+u)} * f(z) |^2 \sigma^*(v)(-z) dz dt \leq \sup_{t \in \mathbb{R}} \| \sigma_t \| \int_{\mathbb{R}^d} [g(f)]^2(z) \sigma^*(v)(-z) dz, \quad (3.6)$$

where

$$g(f)(x) = \left( \int_{-\infty}^{\infty} | \psi_{l(t+u)} * f(x) |^2 dt \right)^{1/2}. \quad (3.7)$$

Now since $\int_{\mathbb{R}^d} \psi_t(x) \, dx = 0$, it is well known that

$$\| g(f) \|_p \leq C_p \| f \|_p \quad \forall 1 < p < \infty \quad (3.8)$$

with constant $C_p$ that depends only on $p$ and the dimension of the underlying space. Thus by (3.6) and Hölder’s inequality we have

$$\| J^l_u(f) \|_p^2 \leq \sup_{t \in \mathbb{R}} \| \sigma_t \| \| g(f) \|_p^2 \| \sigma^*(u) \|_q \leq C_p^2 \sup_{t \in \mathbb{R}} \| \sigma_t \| \| \sigma^*(u) \|_{(p/2)'} \| f \|_p^2. \quad (3.9)$$

Hence our result follows by taking the square root on both sides. The case $p < 2$ follows by duality. \qed

4. Proof of the main theorem. Let $\alpha > 0$, $\Omega \in W_\alpha(n)$. Let $\mathcal{P} = (P_1, \ldots, P_d)$ be a polynomial mapping from $\mathbb{R}^n$ into $\mathbb{R}^d$ with $\deg \mathcal{P} = \max_{1 \leq f \leq d} \deg P_f = N$, where $d \geq 1$ and $n \geq 2$. For $0 \leq l \leq N$ let $N_l, Q_l, v_l^1, \lambda_l^1$, and $L_l$ be as in Section 3.

The first step in our proof is to show that each $\Theta^l_t, l = 1, \ldots, N$ satisfies the hypotheses of Lemma 2.5, that is,

$$\| \Theta^l_t \| \leq C(l), \quad (4.1)$$

$$| (\Theta^l_t)(\xi) | \leq C \frac{M_\alpha}{(\log \left[ c 2^l | L_t(\xi) | \right] )^{1+\alpha}}, \quad (4.2)$$

$$| (\Theta^l_t)(\xi) - (\Theta^{l-1}_t)(\xi) | \leq C 2^l | L_t(\xi) |. \quad (4.3)$$

One can easily see that (4.1) holds trivially. Using the cancellation property of $\Omega$, it is easy to see that (4.3) holds. Thus, we need only to verify (4.2). To see that, we notice that

$$| (\Theta^l_t)(\xi) | \leq \int_{S^{n-1}} | \Omega(\gamma') | \left| \int_0^1 e^{-2\pi i \xi \cdot Q(l^t r \gamma')} \, dr \right| d\sigma(\gamma'). \quad (4.4)$$
Now the quantity $\xi \cdot Q^l(2^tr \gamma')$ can be written in the form
\[ \xi \cdot Q^l(2^tr \gamma') = 2^tr^l \lambda G^l(y') + \xi \cdot R(2^tr \gamma'), \]
where $Q^l$ is a homogeneous polynomial of degree $l$ with $\|G^l\| = 1$, $R$ is a polynomial of degree at most $l - 1$ in the variable $r$,
\[ \lambda = \sum_{j=1}^{N_l} |L_{\alpha_j}(\xi)| \geq N_l |L_1(\xi)| \]
and $\alpha_1, \ldots, \alpha_{N_l}$ are the constants that appeared in Section 2. Thus by van der Corput lemma, we have
\[ \left| \int_0^1 e^{-2\pi i \xi \cdot Q^l(2^tr \gamma')} \, dr \right| \leq C \min \left\{ 1, (2^t |L_l(\xi)| |G^l(y')|)^{-1/l} \right\} \]
and hence
\[ \left| \int_0^1 e^{-2\pi i \xi \cdot Q^l(2^tr \gamma')} \, dr \right| \leq C \left[ \frac{\log |G^l(y')|^{-1}}{(\log [c2^{2lt} |L_l(\xi)|])^{1+\alpha}} \right], \]
where $C$ is a constant independent of $t$ and $\xi$. Since $\Omega \in W_\alpha(n)$, the estimate \((4.2)\) follows.

By Lemma 2.5, there exists a family of measures $\{\nu^l_t : l = 1, \ldots, N, t \in \mathbb{R}\}$ such that
\[ \|\nu^l_t\| \leq C(l), \]
\[ |(\nu^l_t)(\xi)| \leq C \frac{M_\alpha}{(\log [c2^{2lt} |L_l(\xi)|])^{1+\alpha}}, \]
\[ |(\nu^l_t)(\xi)| \leq C 2^{lt} |L_l(\xi)|, \]
\[ g^N_t = \sum_{l=1}^N \nu^l_t. \]

Also by Lemma 2.6 and the definition of $\nu^l_t$ (see the proof of Lemma 2.5), we have
\[ \|(\nu^l) * f\|_p \leq C_{p,l} \|f\|_p \quad \forall 1 < p < \infty. \]

Now one can easily see that
\[ 2^{-t} F_{\rho,t}(x) = g^N_t * f(x) = \sum_{l=1}^N \nu^l_t * f(x). \]

Therefore,
\[ \|M_{\rho} f\|_p \leq \sum_{l=1}^N \|M^l_{\rho} f\|_p, \]
where
\[ M^l_{\rho} f(x) = \left( \int_{-\infty}^{\infty} |\nu^l_t * f(x)|^2 \, dt \right)^{1/2}. \]
Thus to show the boundedness of $M_{\beta}f$, it suffices to show that

$$
\|M_{\beta}^l f\|_p \leq C_{p,l} \|f\|_p
$$

(4.17)

for $p \in ((2 + 2\alpha)/(1 + 2\alpha), 2 + 2\alpha)$, and for all $l = 1, \ldots, N$.

To show (4.17), we proceed as follows: let $\Phi$ and $\psi_t$ be as in Section 3. Then

$$
M_{\beta}^l f(x) = \log 2^l \left( \int_{-\infty}^{\infty} \left| \nu_t^l * \psi_{l(t+u)} * f(x) \right|^2 dt \right)^{1/2}
$$

(4.18)

where

$$
S_{u,l}^l f(x) = \left( \int_{-\infty}^{\infty} \left| \nu_t^l * \psi_{l(t+u)} * f(x) \right|^2 dt \right)^{1/2}.
$$

(4.19)

Now by (4.13) and Theorem 3.1, we have

$$
\|S_{u,l}^l f\|_p \leq C_p \|f\|_p
$$

(4.20)

for all $p \in (1, \infty)$ and for $l = 1, \ldots, N$ which in turn implies that

$$
\int_{-1}^{1} \|S_{u,l}^l f\|_p \, du \leq 2C_p \|f\|_p \quad \forall p \in (1, \infty).
$$

(4.21)

On the other hand, if $u \geq 1$, by the estimate (4.11) we have

$$
\|S_{u,l}^l f\|_2 \leq \sqrt{2\log 2} 2^{l-\theta l u} \|f\|_2.
$$

(4.23)

Thus

$$
\|S_{u,l}^l f\|_2 \leq \sqrt{2\log 2} 2^{l-\theta l u} \|f\|_2.
$$

(4.24)

for all $1 < p < \infty$ and for some $\theta = \theta(p) > 0$. Hence we have

$$
\int_{1}^{\infty} \|S_{u,l}^l f\|_p \, du \leq C_p \|f\|_p \quad \text{for } p \in (1, \infty).
$$

(4.25)
Finally, if \( u < -1 \), by the estimate (4.10) and similar argument as in the case of \( u \leq 1 \), we get

\[
\|S^L_{u} f\|_2 \leq C_1 (|u|)^{-1-\alpha} \|f\|_2.
\]

(4.26)

By interpolating between (4.26) and any \( p \in (1, \infty) \) in (4.20), we get that, if \( p \in ((2 + 2\alpha)/(1 + 2\alpha), 2 + 2\alpha) \) there exists \( \beta > 0 \) such that

\[
\|S^L_{u} f\|_p \leq C_p (|u|)^{-\beta} \|f\|_p,
\]

(4.27)

which implies that

\[
\int_{-\infty}^{-1} \|S^L_{u} f\|_p \, du \leq C_p \|f\|_p
\]

(4.28)

for \( p \in ((2 + 2\alpha)/(1 + 2\alpha), 2 + 2\alpha) \).

Hence by combining (4.18), (4.21), (4.25), and (4.28) we get (4.17).

\[\square\]

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