CONVERGENCE THEOREMS OF THE SEQUENCE OF ITERATES FOR A FINITE FAMILY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. Let E be a uniformly convex Banach space, C a nonempty closed convex subset of E . In this paper, we introduce an iteration scheme with errors in the sense of Xu (1998) generated by $\{T_j : C \rightarrow C\}_{j=1}^r$ as follows: $U_{n(j)} = a_{n(j)}I + b_{n(j)}T_j^n U_{n(j-1)} + c_{n(j)}u_{n(j)}$, $j = 1, 2, \dots, r$, $x_1 \in C$, $x_{n+1} = a_{n(r)}x_n + b_{n(r)}T_r^n U_{n(r-1)}x_n + c_{n(r)}u_{n(r)}$, $n \geq 1$, where $U_{n(0)} := I$, I the identity map; and $\{u_{n(j)}\}$ are bounded sequences in C ; and $\{a_{n(j)}\}$, $\{b_{n(j)}\}$, and $\{c_{n(j)}\}$ are suitable sequences in $[0, 1]$. We first consider the behaviour of iteration scheme above for a finite family of asymptotically nonexpansive mappings. Then we generalize theorems of Schu and Rhoades.

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1. Introduction. Let C be a nonempty convex subset of a Banach space E . A mapping $T : C \rightarrow C$ is called *asymptotically nonexpansive with sequence* $\{k_n\}_{n=1}^\infty$ if $k_n \geq 1$ and $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \quad (1.1)$$

for all $x, y \in C$ and all $n \in \mathbb{N}$. T is called *uniformly L -Lipschitzian* if

$$\|T^n x - T^n y\| \leq L \|x - y\| \quad (1.2)$$

for all $x, y \in C$ and all $n \in \mathbb{N}$. It is clear that every asymptotically nonexpansive mapping is also uniformly L -Lipschitzian for some $L > 0$. In [7], Schu introduced the *modified Ishikawa iteration method* as

$$x_{n+1} = \alpha_n T^n (\beta_n T^n x_n + (1 - \beta_n)x_n) + (1 - \alpha_n)x_n, \quad n = 1, 2, \dots, \quad (1.3)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are suitable sequences in $[0, 1]$ and the *modified Mann iteration method* as

$$x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n)x_n, \quad n = 1, 2, \dots, \quad (1.4)$$

where $\{\alpha_n\}$ is a suitable sequence in $[0, 1]$.

Using the iteration method (1.4), Schu [9, Lemma 1.5] and Rhoades [6, Theorem 1] obtained the following result: *let C be a bounded closed convex subset of a uniformly*

convex Banach space E , $T : C \rightarrow C$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, and $\{\alpha_n\}$ a sequence in $[0, 1]$ satisfying the condition $\varepsilon \leq \alpha_n \leq 1 - \varepsilon$ for all $n \in \mathbb{N}$ and some $\varepsilon > 0$. Suppose that $x_1 \in C$ and that $\{x_n\}$ is given by (1.4). Then $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$.

Note that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ if and only if $\sum_{n=1}^{\infty} (k_n^s - 1) < \infty$ for some $s > 1$ (see [5, Remark 3]).

Let C be a nonempty convex subset of a Banach space E . Let $T_j : C \rightarrow C$ be a given mapping for each $j = 1, 2, \dots, r$. We now introduce an iteration scheme with errors in the sense of Xu [11] generated by $T_1, T_2, \dots, T_r$ as follows: let $U_{n(0)} = I$, where I is the identity map,

$$\begin{aligned} U_{n(1)} &= a_{n(1)}I + b_{n(1)}T_1^n U_{n(0)} + c_{n(1)}u_{n(1)}, \\ U_{n(2)} &= a_{n(2)}I + b_{n(2)}T_2^n U_{n(1)} + c_{n(2)}u_{n(2)}, \\ &\vdots \\ U_{n(r)} &= a_{n(r)}I + b_{n(r)}T_r^n U_{n(r-1)} + c_{n(r)}u_{n(r)}, \\ x_1 &\in C, \quad x_{n+1} = a_{n(r)}x_n + b_{n(r)}T_r^n U_{n(r-1)}x_n + c_{n(r)}u_{n(r)}, \quad n \geq 1. \end{aligned} \tag{1.5}$$

Here, $\{u_{n(j)}\}_{n=1}^{\infty}$ is a bounded sequence in C for each $j = 1, 2, \dots, r$, and $\{a_{n(j)}\}_{n=1}^{\infty}$, $\{b_{n(j)}\}_{n=1}^{\infty}$, and $\{c_{n(j)}\}_{n=1}^{\infty}$ are sequences in $[0, 1]$ satisfying the conditions

$$a_{n(j)} + b_{n(j)} + c_{n(j)} = 1 \tag{1.6}$$

for all $n \in \mathbb{N}$ and each $j = 1, 2, \dots, r$. This scheme contains the modified Mann and Ishikawa iteration methods with errors in the sense of Xu [11] (cf. [5]): for $r = 1$, our scheme reduces to Mann-Xu type iteration and for $r = 2$, $T_1 = T_2$ to Ishikawa-Xu type iteration.

In 1972, Goebel and Kirk [1] proved that if C is a bounded closed convex subset of a uniformly convex Banach space E , then every asymptotically nonexpansive selfmapping T of C has a fixed point. After the existence theorem of Goebel and Kirk [1], several authors including Schu [7, 9], Rhoades [6], Huang [3] and Osilike and Aniagbosor [5] have studied methods for the iterative approximation of fixed points of asymptotically nonexpansive mappings. In this paper, we first extend the result above of [9, Lemma 1.5] and [6, Theorem 1] to the iteration scheme (1.5) and without the restrictions that C is bounded. Then, using this result, we generalize [9, Theorems 2.1, 2.2, and 2.4] and [6, Theorems 2 and 3].

In the sequel, we will need the following results.

LEMMA 1.1 (see [5, Lemma 1]). *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 1. \tag{1.7}$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists. In particular, if $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim_{n \rightarrow \infty} a_n = 0$.

LEMMA 1.2 (see [8, Lemma 2]). *Let $\{\beta_n\}_{n=1}^\infty$ and $\{\omega_n\}_{n=1}^\infty$ be sequences of nonnegative numbers such that for some real numbers $N_0 \geq 1$,*

$$\beta_{n+1} \leq (1 - \delta_n)\beta_n + \omega_n \quad (1.8)$$

for all $n \geq N_0$, where $\delta_n \in [0, 1]$. If $\sum_{n=1}^\infty \delta_n = \infty$ and $\sum_{n=1}^\infty \omega_n < \infty$, then $\lim_{n \rightarrow \infty} \beta_n = 0$.

THEOREM 1.3 (see [10, Theorem 2]). *Let E be a uniformly convex Banach space and $r > 0$. Then there exists a continuous, strictly increasing and convex function $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|) \quad (1.9)$$

for all $x, y \in B_r := \{x \in E : \|x\| \leq r\}$ and $\lambda \in [0, 1]$.

A Banach space E is said to satisfy Opial's condition [4] if $x_n \rightarrow x$ weakly and $x \neq y$ imply

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|. \quad (1.10)$$

LEMMA 1.4 (see [2, Lemma 4]). *Let E be a uniformly convex Banach space satisfying Opial's condition and C a nonempty closed convex subset of E . Let $T: C \rightarrow C$ be an asymptotically nonexpansive mapping. Then $(I - T)$ is demiclosed at zero, that is, for each sequence $\{x_n\}$ in C , the conditions $x_n \rightarrow x$ weakly and $(I - T)x_n \rightarrow 0$ strongly imply $(I - T)x = 0$.*

2. Main results. For abbreviation, we denote the set of fixed points of a mapping T by $F(T)$, and now prove the following results.

THEOREM 2.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T_j: C \rightarrow C$ an asymptotically nonexpansive mapping with sequence $\{k_{n(j)}\}_{n=1}^\infty$ for each $j = 1, 2, \dots, r$ such that $\sum_{n=1}^\infty (k_n - 1) < \infty$, where $k_n := \max_{1 \leq j \leq r} \{k_{n(j)}\} \geq 1$ and $\cap_{j=1}^r F(T_j) \neq \emptyset$. Let $\{u_{n(j)}\}_{n=1}^\infty$ be a bounded sequence in C for each $j = 1, 2, \dots, r$ and let $\{a_{n(j)}\}_{n=1}^\infty$, $\{b_{n(j)}\}_{n=1}^\infty$, and $\{c_{n(j)}\}_{n=1}^\infty$ be sequences in $[0, 1]$ satisfying the conditions:*

- (i) $a_{n(j)} + b_{n(j)} + c_{n(j)} = 1$ for all $n \in \mathbb{N}$ and each $j = 1, 2, \dots, r$;
- (ii) $\sum_{n=1}^\infty c_{n(j)} < \infty$ for each $j = 1, 2, \dots, r$;
- (iii) $0 < a \leq \alpha_{n(j)} \leq b < 1$ for all $n \in \mathbb{N}$, each $j = 1, 2, \dots, r$, and some constants a, b , where $\alpha_{n(j)} := b_{n(j)} + c_{n(j)}$.

Suppose that $\{x_n\}$ is given by (1.5). Then $\lim_{n \rightarrow \infty} \|x_n - T_j x_n\| = 0$ for each $j = 1, 2, \dots, r$.

In order to prove Theorem 2.1, we first prove the following lemmas.

LEMMA 2.2. *Let C be a nonempty convex subset of a Banach space E . Let $T_j: C \rightarrow C$ be a uniformly L -Lipschitzian mapping for each $j = 1, 2, \dots, r$, and let $\{x_n\}$ be as in (1.5).*

Set $e_{n(j)} := \|x_n - T_j^n U_{n(j-1)} x_n\|$ for all $n, j \in \mathbb{N}$. Then for all $n \geq 2$,

$$\begin{aligned} \|x_n - T_1 x_n\| &\leq e_{n(1)} + (L^2 + L)e_{n-1(r)} + L e_{n-1(1)} + (L^2 + L)c_{n-1(r)} \|u_{n-1(r)} - x_{n-1}\|, \\ \|x_n - T_j x_n\| &\leq e_{n(j)} + (L^2 + L)e_{n-1(r)} + L^2 e_{n(j-1)} + L^2 e_{n-1(j-1)} + L e_{n-1(j)} \\ &\quad + (L^2 + L)c_{n-1(r)} \|u_{n-1(r)} - x_{n-1}\| + L^2 c_{n(j-1)} \|u_{n(j-1)} - x_n\| \\ &\quad + L^2 c_{n-1(j-1)} \|x_{n-1} - u_{n-1(j-1)}\|, \end{aligned} \quad (2.1)$$

for each $j = 2, 3, \dots, r$.

PROOF. Observe that for $j = 2, 3, \dots, r$ we have

$$\begin{aligned} &\|U_{n(j-1)} x_n - U_{n-1(j-1)} x_{n-1}\| \\ &= \|(a_{n(j-1)} x_n + b_{n(j-1)} T_{j-1}^n U_{n(j-2)} x_n + c_{n(j-1)} u_{n(j-1)}) \\ &\quad - (a_{n-1(j-1)} x_{n-1} + b_{n-1(j-1)} T_{j-1}^{n-1} U_{n-1(j-2)} x_{n-1} \\ &\quad + c_{n-1(j-1)} u_{n-1(j-1)})\| \\ &= \|(x_n - x_{n-1}) + b_{n(j-1)} (T_{j-1}^n U_{n(j-2)} x_n - x_n) \\ &\quad + c_{n(j-1)} (u_{n(j-1)} - x_n) + b_{n-1(j-1)} (x_{n-1} - T_{j-1}^{n-1} U_{n-1(j-2)} x_{n-1}) \\ &\quad + c_{n-1(j-1)} (x_{n-1} - u_{n-1(j-1)})\| \\ &\leq \|x_n - x_{n-1}\| + e_{n(j-1)} + e_{n-1(j-1)} + c_{n(j-1)} \|u_{n(j-1)} - x_n\| \\ &\quad + c_{n-1(j-1)} \|x_{n-1} - u_{n-1(j-1)}\|, \end{aligned} \quad (2.2)$$

$$\begin{aligned} \|x_n - x_{n-1}\| &= \|a_{n-1(r)} x_{n-1} + b_{n-1(r)} T_r^{n-1} U_{n-1(r-1)} x_{n-1} + c_{n-1(r)} u_{n-1(r)} - x_{n-1}\| \\ &\leq b_{n-1(r)} \|T_r^{n-1} U_{n-1(r-1)} x_{n-1} - x_{n-1}\| + c_{n-1(r)} \|u_{n-1(r)} - x_{n-1}\| \\ &\leq e_{n-1(r)} + c_{n-1(r)} \|u_{n-1(r)} - x_{n-1}\|. \end{aligned} \quad (2.3)$$

Therefore,

$$\begin{aligned} \|x_n - T_j x_n\| &\leq \|x_n - T_j^n U_{n(j-1)} x_n\| + \|T_j^n U_{n(j-1)} x_n - T_j x_n\| \\ &\leq e_{n(j)} + L \|T_j^{n-1} U_{n(j-1)} x_n - x_n\| \\ &\leq e_{n(j)} + L \|T_j^{n-1} U_{n(j-1)} x_n - T_j^{n-1} U_{n-1(j-1)} x_{n-1}\| \\ &\quad + L \|T_j^{n-1} U_{n-1(j-1)} x_{n-1} - x_{n-1}\| + L \|x_{n-1} - x_n\| \\ &\leq e_{n(j)} + L^2 \|U_{n(j-1)} x_n - U_{n-1(j-1)} x_{n-1}\| \\ &\quad + L e_{n-1(j)} + L \|x_{n-1} - x_n\|. \end{aligned} \quad (2.4)$$

Using (2.3) in (2.4) for $j = 1$ we have

$$\begin{aligned} \|x_n - T_1 x_n\| &\leq e_{n(1)} + (L^2 + L) \|x_n - x_{n-1}\| + L e_{n-1(1)} \\ &\leq e_{n(1)} + (L^2 + L) e_{n-1(r)} + L e_{n-1(1)} \\ &\quad + (L^2 + L) c_{n-1(r)} \|U_{n-1(r)} - x_{n-1}\|. \end{aligned} \quad (2.5)$$

Using (2.2) and (2.3) in (2.4) for $j = 2, 3, \dots, r$ we have

$$\begin{aligned}
 \|x_n - T_j x_n\| &\leq e_{n(j)} + (L^2 + L)\|x_n - x_{n-1}\| + L^2 e_{n(j-1)} + L^2 e_{n-1(j-1)} + L e_{n-1(j)} \\
 &\quad + L^2 c_{n(j-1)}\|u_{n(j-1)} - x_n\| + L^2 c_{n-1(j-1)}\|x_{n-1} - u_{n-1(j-1)}\| \\
 &\leq e_{n(j)} + (L^2 + L)e_{n-1(r)} + L^2 e_{n(j-1)} + L^2 e_{n-1(j-1)} + L e_{n-1(j)} \\
 &\quad + (L^2 + L)c_{n-1(r)}\|u_{n-1(r)} - x_{n-1}\| + L^2 c_{n(j-1)}\|u_{n(j-1)} - x_n\| \\
 &\quad + L^2 c_{n-1(j-1)}\|x_{n-1} - u_{n-1(j-1)}\|.
 \end{aligned} \tag{2.6}$$

This completes the proof of Lemma 2.2. $\square$

LEMMA 2.3. Let C be a nonempty convex subset of a Banach space E . Let $\{T_1, T_2, \dots, T_r\}$, $\{u_{n(j)}\}$, and $\{x_n\}$ be as in Theorem 2.1 and let $\{a_{n(j)}\}$, $\{b_{n(j)}\}$, and $\{c_{n(j)}\}$ satisfy conditions (i) and (ii) of Theorem 2.1. Then $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists for all $x^* \in \cap_{j=1}^r F(T_j)$.

PROOF. Let $x^* \in \cap_{j=1}^r F(T_j)$. Since $\{u_{n(j)}\}_{n=1}^\infty$ and $\{k_n\}_{n=1}^\infty$ are bounded, there exists a constant $N > 0$ such that $\sup_{n \in \mathbb{N}} \{\|u_{n(j)} - x^*\| : j = 1, 2, \dots, r\} \leq N$ and $\sup_{n \in \mathbb{N}} \{1 + k_n + \dots + k_n^{r-1}\} \leq N$. Then, we have

$$\begin{aligned}
 \|x_{n+1} - x^*\| &= \|a_{n(r)}x_n + b_{n(r)}T_r^n U_{n(r-1)}x_n + c_{n(r)}u_{n(r)} - x^*\| \\
 &\leq a_{n(r)}\|x_n - x^*\| + b_{n(r)}\|T_r^n U_{n(r-1)}x_n - x^*\| + c_{n(r)}\|u_{n(r)} - x^*\| \\
 &\leq a_{n(r)}\|x_n - x^*\| + b_{n(r)}k_n\|U_{n(r-1)}x_n - x^*\| + Nc_{n(r)} \\
 &= a_{n(r)}\|x_n - x^*\| + b_{n(r)}k_n \\
 &\quad \times \|a_{n(r-1)}(x_n - x^*) + b_{n(r-1)}(T_{r-1}^n U_{n(r-2)}x_n - x^*) \\
 &\quad + c_{n(r-1)}(u_{n(r-1)} - x^*)\| + Nc_{n(r)} \\
 &\leq [1 - b_{n(r)} + (1 - b_{n(r-1)})b_{n(r)}k_n]\|x_n - x^*\| \\
 &\quad + b_{n(r)}b_{n(r-1)}k_n^2\|U_{n(r-2)}x_n - x^*\| + Nc_{n(r)} + N^2c_{n(r-1)} \\
 &\vdots \\
 &\leq [1 - b_{n(r)} + (1 - b_{n(r-1)})b_{n(r)}k_n \\
 &\quad + \dots + (1 - b_{n(1)})b_{n(r)}b_{n(r-1)} \dots b_{n(2)}k_n^{r-1} + b_{n(r)}b_{n(r-1)} \dots b_{n(1)}k_n^r] \\
 &\quad \cdot \|x_n - x^*\| + N(c_{n(r)} + Nc_{n(r-1)} + \dots + Nc_{n(1)}) \\
 &= [1 + b_{n(r)}(k_n - 1) + b_{n(r)}b_{n(r-1)}k_n(k_n - 1) \\
 &\quad + \dots + b_{n(r)}b_{n(r-1)} \dots b_{n(1)}(k_n^{r-1})(k_n - 1)]\|x_n - x^*\| + \psi_n \\
 &\leq [1 + (k_n - 1)(1 + k_n + \dots + k_n^{r-1})]\|x_n - x^*\| + \psi_n \\
 &\leq [1 + (k_n - 1)N]\|x_n - x^*\| + \psi_n \\
 &= (1 + \varphi_n)\|x_n - x^*\| + \psi_n,
 \end{aligned} \tag{2.7}$$

for all $n \in \mathbb{N}$, where $\varphi_n := (k_n - 1)N$ and $\psi_n := N(c_{n(r)} + Nc_{n(r-1)} + \cdots + Nc_{n(1)})$. Since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} c_{n(j)} < \infty$ for each $j = 1, 2, \dots, r$, we have $\sum_{n=1}^{\infty} \varphi_n < \infty$ and $\sum_{n=1}^{\infty} \psi_n < \infty$. Thus, $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists by Lemma 1.1. This completes the proof of Lemma 2.3. $\square$

LEMMA 2.4. *Under the hypotheses of Lemma 2.3, if E is a uniformly convex Banach space, then there exists a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $g(0) = 0$, and*

$$\sum_{n=1}^{\infty} \left[\sum_{j=1}^r \left(\prod_{l=j}^r \alpha_{n(l)} \right) (1 - \alpha_{n(j)}) g(\|x_n - T_j^n U_{n(j-1)} x_n\|) \right] < \infty, \quad (2.8)$$

where $\alpha_{n(j)} := b_{n(j)} + c_{n(j)}$ for all $n \in \mathbb{N}$ and each $j = 1, 2, \dots, r$.

PROOF. Let $x^* \in \cap_{j=1}^r F(T_j)$. Lemma 2.3 and the hypotheses of Lemma 2.4 imply that $\{x_n - x^*\}_{n=1}^{\infty}$, $\{u_{n(j)}\}_{n=1}^{\infty}$, and $\{k_n\}_{n=1}^{\infty}$ are bounded. Then, there exists a constant $d > 0$ such that

$$\cup_{j=1}^r \{T_j^n U_{n(j-1)} x_n - x^*\}_{n=1}^{\infty} \cup \{x_n - x^*\}_{n=1}^{\infty} \subseteq B_d. \quad (2.9)$$

By Theorem 1.3, there exists a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $g(0) = 0$, and

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|) \quad (2.10)$$

for all $x, y \in B_d$ and $\lambda \in [0, 1]$. By inequality (2.10) we obtain the following estimate: for some constant M , we have

$$\begin{aligned} \|U_{n(j)} x_n - x^*\|^2 &= \|(1 - \alpha_{n(j)})(x_n - x^*) + \alpha_{n(j)}(T_j^n U_{n(j-1)} x_n - x^*) \\ &\quad - c_{n(j)}(T_j^n U_{n(j-1)} x_n - u_{n(j)})\|^2 \\ &\leq \left(\|(1 - \alpha_{n(j)})(x_n - x^*) + \alpha_{n(j)}(T_j^n U_{n(j-1)} x_n - x^*)\| \right. \\ &\quad \left. + c_{n(j)} \|(T_j^n U_{n(j-1)} x_n - u_{n(j)})\| \right)^2 \\ &\leq \|(1 - \alpha_{n(j)})(x_n - x^*) + \alpha_{n(j)}(T_j^n U_{n(j-1)} x_n - x^*)\|^2 + c_{n(j)} M \quad (2.11) \\ &\leq (1 - \alpha_{n(j)}) \|x_n - x^*\|^2 + \alpha_{n(j)} \|T_j^n U_{n(j-1)} x_n - x^*\|^2 \\ &\quad - \alpha_{n(j)} (1 - \alpha_{n(j)}) g(\|x_n - T_j^n U_{n(j-1)} x_n\|) + c_{n(j)} M \\ &\leq (1 - \alpha_{n(j)}) \|x_n - x^*\|^2 + \alpha_{n(j)} k_n^2 \|U_{n(j-1)} x_n - x^*\|^2 \\ &\quad - \alpha_{n(j)} (1 - \alpha_{n(j)}) g(\|x_n - T_j^n U_{n(j-1)} x_n\|) + c_{n(j)} M, \end{aligned}$$

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_{n(r)})(x_n - x^*) + \alpha_{n(r)}(T_r^n U_{n(r-1)} x_n - x^*) \\ &\quad - c_{n(r)}(T_r^n U_{n(r-1)} x_n - u_{n(r)})\|^2 \\ &\leq (1 - \alpha_{n(r)}) \|x_n - x^*\|^2 + \alpha_{n(r)} k_n^2 \|U_{n(r-1)} x_n - x^*\|^2 \\ &\quad - \alpha_{n(r)} (1 - \alpha_{n(r)}) g(\|x_n - T_r^n U_{n(r-1)} x_n\|) + c_{n(r)} M. \end{aligned} \quad (2.12)$$

By a repeated application of inequality (2.11) in (2.12), we obtain

$$\begin{aligned}
 \|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 \\
 &\quad + \alpha_{n(r)}(k_n^2 - 1)(1 + \alpha_{n(r-1)}k_n^2 + \cdots \\
 &\quad \quad + \alpha_{n(r-1)}\alpha_{n(r-2)}\cdots\alpha_{n(1)}k_n^{2(r-1)})\|x_n - x^*\|^2 \\
 &\quad - \sum_{j=1}^r \left(\prod_{l=j}^r \alpha_{n(l)} \right) (1 - \alpha_{n(j)}) \mathcal{G}(\|x_n - T_j^n U_{n(j-1)} x_n\|) \\
 &\quad + (c_{n(r)} + k_n^2 c_{n(r-1)} + \cdots + k_n^{2(r-1)} c_{n(1)})M.
 \end{aligned} \tag{2.13}$$

Since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, hence $\lim_{n \rightarrow \infty} k_n = 1$, we may assume that $k_n \leq L$ for all $n \in \mathbb{N}$ and some constant L . Let $N = \max_{1 \leq j \leq r} \{L^{2j}\} \geq 1$. Then

$$\begin{aligned}
 \|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 + (k_n - 1)(N + 1)rNd^2 + MN \sum_{j=1}^r c_{n(j)} \\
 &\quad - \sum_{j=1}^r \left(\prod_{l=j}^r \alpha_{n(l)} \right) (1 - \alpha_{n(j)}) \mathcal{G}(\|x_n - T_j^n U_{n(j-1)} x_n\|)
 \end{aligned} \tag{2.14}$$

for all $n \in \mathbb{N}$. Transposing and summing from 1 to m we have

$$\begin{aligned}
 \sum_{n=1}^m \left[\sum_{j=1}^r \left(\prod_{l=j}^r \alpha_{n(l)} \right) (1 - \alpha_{n(j)}) \mathcal{G}(\|x_n - T_j^n U_{n(j-1)} x_n\|) \right] \\
 \leq \|x_1 - x^*\|^2 + (N + 1)rNd^2 \sum_{n=1}^m (k_n - 1) + MN \sum_{n=1}^m \sum_{j=1}^r c_{n(j)}.
 \end{aligned} \tag{2.15}$$

Since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} c_{n(j)} < \infty$ for each $j = 1, 2, \dots, r$, it follows that

$$\sum_{n=1}^{\infty} \left[\sum_{j=1}^r \left(\prod_{l=j}^r \alpha_{n(l)} \right) (1 - \alpha_{n(j)}) \mathcal{G}(\|x_n - T_j^n U_{n(j-1)} x_n\|) \right] < \infty. \tag{2.16}$$

This completes the proof of Lemma 2.4. $\square$

We now give the proof of Theorem 2.1.

PROOF OF THEOREM 2.1. By Lemma 2.4 and condition (iii), we have

$$(1 - b) \sum_{n=1}^{\infty} \sum_{j=1}^r a^{r-j+1} \mathcal{G}(\|x_n - T_j^n U_{n(j-1)} x_n\|) < \infty. \tag{2.17}$$

Thus,

$$\sum_{j=1}^r \mathcal{G}(\|x_n - T_j^n U_{n(j-1)} x_n\|) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{2.18}$$

Since $\mathcal{G}$ is a continuous and strictly increasing function with $\mathcal{G}(0) = 0$, we have $\lim_{n \rightarrow \infty} \|x_n - T_j^n U_{n(j-1)} x_n\| = 0$ for each $j = 1, 2, \dots, r$. Since $\{x_n - x^*\}$ and $\{u_{n(j)}\}$ are bounded. So we have

$$\sup_{n \in \mathbb{N}} \{\|x_n - u_{n(j)}\| : j = 1, 2, \dots, r\} \leq D \tag{2.19}$$

for some constant $D > 0$. Let $e_{n(j)} = \|x_n - T_j^n U_{n(j-1)} x_n\|$ and L be as in the proof of Lemma 2.4. Then, by Lemma 2.2, we have

$$\begin{aligned} \|x_n - T_1 x_n\| &\leq e_{n(1)} + (L^2 + L)e_{n-1(r)} + Le_{n-1(1)} + (L^2 + L)c_{n-1(r)}D \rightarrow 0 \quad \text{as } n \rightarrow \infty, \\ \|x_n - T_j x_n\| &\leq e_{n(j)} + (L^2 + L)e_{n-1(r)} + L^2 e_{n(j-1)} + L^2 e_{n-1(j-1)} + Le_{n-1(j)} \\ &\quad + (L^2 + L)c_{n-1(r)}D + L^2 c_{n(j-1)}D + L^2 c_{n-1(j-1)}D \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned} \quad (2.20)$$

for each $j = 2, 3, \dots, r$. This completes the proof of Theorem 2.1. $\square$

THEOREM 2.5. *Under the hypotheses of Theorem 2.1, if E is a uniformly convex Banach space satisfying Opial's condition, then $\{x_n\}$ converges weakly to a common fixed point of $T_1, T_2, \dots, T_r$.*

PROOF. Let $\omega_w(\{x_n\})$ be the set of all weak subsequential limits of a bounded sequence $\{x_n\}$ in C . By Lemma 1.4 and Theorem 2.1, $\omega_w(\{x_n\})$ is contained in $\cap_{j=1}^r F(T_j)$.

The remainder of the proof is similar to that of [9, Theorem 2.1], so the details are omitted. $\square$

REMARK 2.6. Theorem 2.5 generalizes [9, Theorem 2.1].

THEOREM 2.7. *Under the hypotheses of Theorem 2.1. Suppose that T_1^m is compact for some $m \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \dots, T_r$.*

PROOF. As in the proof of [9, Theorem 2.2] by using Theorem 2.1 and Lemma 2.3, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ such that $\lim_{i \rightarrow \infty} x_{n_i} = p$. Thus, by Theorem 2.1, we obtain that $T_j p = p$ for each $j = 1, 2, \dots, r$. Hence, $p \in \cap_{j=1}^r F(T_j)$ and it follows from Lemma 2.3 that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Therefore, we conclude that $\lim_{n \rightarrow \infty} \|x_n - p\| = 0$, completing the proof of Theorem 2.7. $\square$

REMARK 2.8. Theorem 2.7 generalizes [9, Theorem 2.2] and [6, Theorems 2 and 3].

LEMMA 2.9. *Let K be a compact convex subset of a normed space E . Suppose that $\alpha, \beta, \gamma \in [0, 1]$ such that $\alpha + \beta + \gamma = 1$. Then*

$$d(\alpha x + \beta y + \gamma z, K) \leq \alpha d(x, K) + \beta d(y, K) + \gamma d(z, K) \quad (2.21)$$

for all $x, y, z \in E$ where $d(x, K) := \inf\{\|x - p\| : p \in K\}$.

PROOF. Let $x, y, z \in E$. Since K is compact, we have $d(x, p_1) = d(x, K)$, $d(y, p_2) = d(y, K)$, and $d(z, p_3) = d(z, K)$ for some $p_1, p_2, p_3 \in K$. Since K is convex so that $\alpha p_1 + \beta p_2 + \gamma p_3 \in K$. Therefore,

$$\begin{aligned} d(\alpha x + \beta y + \gamma z, K) &\leq \|(\alpha x + \beta y + \gamma z) - (\alpha p_1 + \beta p_2 + \gamma p_3)\| \\ &\leq \alpha \|x - p_1\| + \beta \|y - p_2\| + \gamma \|z - p_3\| \\ &= \alpha d(x, K) + \beta d(y, K) + \gamma d(z, K). \end{aligned} \quad (2.22)$$

This completes the proof of Lemma 2.9. $\square$

THEOREM 2.10. *Under the hypotheses of Theorem 2.1. Suppose that there exists a nonempty compact convex subset K of E and some $\alpha \in (0, 1)$ such that $d(T_j x, K) \leq \alpha d(x, K)$ for all $x \in C$ and each $j = 1, 2, \dots, r$. Then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \dots, T_r$.*

PROOF. For $n \in \mathbb{N}$ and $x \in C$ we have $d(T_j^n x, K) \leq \alpha^n d(x, K)$ for each $j = 1, 2, \dots, r$. Since $\{u_{n(j)}\}_{n=1}^\infty$ is bounded for each $j = 1, 2, \dots, r$ and K is compact. Thus, there exists a constant $D > 0$ such that

$$\sup_{n \in \mathbb{N}} \{d(u_{n(j)}, K) : j = 1, 2, \dots, r\} \leq D. \quad (2.23)$$

Then, by Lemma 2.9, we have

$$\begin{aligned} d(x_{n+1}, K) &= d(a_{n(r)}x_n + b_{n(r)}T_r^n U_{n(r-1)}x_n + c_{n(r)}u_{n(r)}, K) \\ &\leq a_{n(r)}d(x_n, K) + b_{n(r)}d(T_r^n U_{n(r-1)}x_n, K) + c_{n(r)}d(u_{n(r)}, K) \\ &\leq a_{n(r)}d(x_n, K) + b_{n(r)}\alpha^n d(U_{n(r-1)}x_n, K) + c_{n(r)}D \\ &\leq (1 - b_{n(r)})d(x_n, K) + b_{n(r)}\alpha^n d(a_{n(r-1)}x_n + b_{n(r-1)}T_{r-1}^n U_{n(r-2)}x_n \\ &\quad + c_{n(r-1)}u_{n(r-1)}, K) + c_{n(r)}D \\ &\leq [1 - b_{n(r)} + (1 - b_{n(r-1)})b_{n(r)}\alpha^n]d(x_n, K) \\ &\quad + b_{n(r)}b_{n(r-1)}\alpha^{2n}d(U_{n(r-2)}x_n, K) + (c_{n(r-1)} + c_{n(r)})D \\ &\vdots \\ &\leq [1 - b_{n(r)}(1 - \alpha^n)(1 + b_{n(r-1)}\alpha^n + \dots + b_{n(r-1)}b_{n(r-2)}\dots b_{n(1)}\alpha^{(r-1)n})] \\ &\quad d(x_n, K) + (c_{n(1)} + c_{n(2)} + \dots + c_{n(r)})D \\ &\leq [1 - a(1 - \alpha^n)(1 + a\alpha^n + \dots + a^{r-1}\alpha^{(r-1)n})]d(x_n, K) + D \sum_{j=1}^r c_{n(j)}. \end{aligned} \quad (2.24)$$

Let $\delta_n = a(1 - \alpha^n)(1 + a\alpha^n + \dots + a^{r-1}\alpha^{(r-1)n})$. Since $\lim_{n \rightarrow \infty} \delta_n = a$ and $0 < a < 1$, then there exists a real number $N_0 \geq 1$ such that $\delta_n < 1$ for all $n \geq N_0$. Since $\sum_{n=1}^\infty \delta_n = \infty$ and $\sum_{n=1}^\infty \sum_{j=1}^r c_{n(j)} < \infty$, then by Lemma 1.2, we have $\lim_{n \rightarrow \infty} d(x_n, K) = 0$. Since K is compact, this is easily seen to imply that $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ such that $\lim_{i \rightarrow \infty} x_{n_i} = p$. The rest of the proof is identical to the related part of the proof of Theorem 2.7. $\square$

REMARK 2.11. Theorem 2.10 generalizes [9, Theorem 2.4].

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REFERENCES

- [1] K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171–174. [MR 45#7552](#). [Zbl 256.47045](#).

- [2] J. Górnicki, *Weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces*, Comment. Math. Univ. Carolin. **30** (1989), no. 2, 249–252. [MR 90g:47097](#). [Zbl 686.47045](#).
- [3] Z. Huang, *Mann and Ishikawa iterations with errors for asymptotically nonexpansive mappings*, Comput. Math. Appl. **37** (1999), no. 3, 1–7. [MR 2000a:47118](#). [Zbl 942.47046](#).
- [4] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597. [MR 35#2183](#). [Zbl 179.19902](#).
- [5] M. O. Osilike and S. C. Aniagbosor, *Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings*, Math. Comput. Modelling **32** (2000), no. 10, 1181–1191. [MR 2001h:47089](#). [Zbl 01581388](#).
- [6] B. E. Rhoades, *Fixed point iterations for certain nonlinear mappings*, J. Math. Anal. Appl. **183** (1994), no. 1, 118–120. [MR 95e:47082](#). [Zbl 807.47045](#).
- [7] J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **158** (1991), no. 2, 407–413. [MR 92d:47072](#). [Zbl 734.47036](#).
- [8] ———, *On a theorem of C. E. Chidume concerning the iterative approximation of fixed points*, Math. Nachr. **153** (1991), 313–319. [MR 93b:47123](#). [Zbl 796.47047](#).
- [9] ———, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. **43** (1991), no. 1, 153–159. [MR 91k:47136](#). [Zbl 709.47051](#).
- [10] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1991), no. 12, 1127–1138. [MR 92e:47126](#). [Zbl 757.46033](#).
- [11] Y. Xu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl. **224** (1998), no. 1, 91–101. [MR 99g:47144](#). [Zbl 936.47041](#).

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