IJMMS 27:3 (2001) 161–167
PII. S0161171201010754
http://ijmms.hindawi.com


# FACTORIZATION OF GROUPS INVOLVING SYMMETRIC AND ALTERNATING GROUPS

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(Received 16 August 2000 and in revised form 28 September 2000)

ABSTRACT. We obtain the structure of finite groups of the form $G = AB$ where $B$ is a group isomorphic to the symmetric group on $n$ letters $S_n$, $n \geq 5$ and $A$ is a group isomorphic to the alternating group on 6 letters.

2000 Mathematics Subject Classification. 20D40.

**1. Introduction.** For a finite group $G$ if there exist subgroups $A$ and $B$ of $G$ such that $G = AB$, then $G$ is called a factorizable group. Of course if neither of $A$ nor $B$ is contained in the other, then the factorization is called nontrivial. A knowledge of the factorizations of finite simple groups will help to investigate the general theory of factorizing finite groups. All possible factorizations of sporadic simple groups have been obtained in [4] and those of simple groups of Lie type of Lie rank 1 or 2 in [3].

A factorization $G = AB$ where both $A$ and $B$ are maximal subgroups of $G$ is called a maximal factorization of $G$. In [9], all the maximal factorizations of all the finite simple groups and their automorphism groups have been determined completely.

In another direction some results have been obtained assuming $G = AB$ is a factorization of $G$ with $A$ and $B$ simple subgroups of $G$. For example, in [8] finite groups $G = AB$ where both $A$ and $B$ are isomorphic to the simple group of order 60 are classified, and in [10] finite groups $G = AB$ where $A$ is a non-abelian simple group and $B \cong A_5$ are determined. In [5], $G = AB$ where $A$ and $B$ are simple groups of small order are considered.

In a series of papers, Walls considered groups which are a product of simple groups [13, 14]. In [15], groups which are product of a symmetric group and a group isomorphic to $A_5$ are classified. This result is interesting because in the factorization $G = AB$ one of the factors is not a simple group. Motivated by this result, in this paper we classify all groups $G$ which are product of subgroups $A$ and $B$ such that $A \cong A_6$ and $B \cong S_n$, $n \geq 6$. In this paper, $A_n$ and $S_n$ are the alternating and symmetric groups on $n$ letters, respectively, and all groups are assumed to be finite.

**2. Preliminary results.** Now $A_6$ is a simple group of order 360 and it is easy to verify that the order of any proper subgroup of $A_6$ is one of the numbers 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 18, 24, 36, or 60. Therefore the size of sets on which $A_6$ acts transitively and faithfully is one of the numbers 360, 180, 120, 90, 72, 60, 45, 40, 36, 30, 20, 15, 10, or 6. Also since $A_6 \cong L_2(9)$, $A_6$ has a 2-transitive action on a set of 10 points and by

consulting [10] one can see that if $A_6$ acts $k$-transitively, $k \geq 2$, on a set of cardinality $m$, then either $m = 10$, $k = 2$ or $m = 6$ and $k = 2, 3$, or 4. Now by [14, Lemma 7] we have the following decomposition.

**Lemma 2.1.** *For $n$ a positive integer, $S_{n+1} = A_6 S_n$ and $A_{n+1} = A_6 A_n$ if and only if $n = 5, 9, 14, 19, 29, 35, 39, 44, 59, 71, 89, 119, 179$, or $359$. We can write $A_{10} = AB$ where $A \cong A_6$ and $B \cong A_8$. Further, we can write $S_{10} = AB$ and $A_{10} \times Z_2 = AB$ where $A \cong A_6$ and $B \cong S_8$.*

The only nontrivial decomposition $A_m = AB$, where $A \cong A_6$ and $B \cong A_n$, occurs if and only if $m = n + 1$, where $n$ is one of the numbers mentioned in Lemma 2.1 or $m = 10$ and $n = 8$. To see this one can use [14, Theorem 9]. Because according to this theorem one of the groups in the decomposition $G = AB$, say $A$, must be a $k$-transitive permutation group and according to what we said earlier all the $k$-transitive permutation representations of $A \cong A_6$ are known.

For our work it is necessary to know if it is possible to decompose an alternating group as the product of $A_6$ and $S_n$.

**Lemma 2.2.** *It is not possible to decompose the alternating group $A_m$, $m \geq 7$, as the product of $A_6$ and a symmetric group $S_n$, $n > 1$, unless $m = 10$ and $n = 8$.*

**Proof.** According to [9, Theorem D], if $A_m$ acts naturally on a set $\Omega$ of cardinality $m$, and $A_m = A_6 S_n$, then there are two possibilities.

**Case (i).** $A_{m-k} \trianglelefteq S_n \leq S_{m-k} \times S_k$ for some $k$, $1 \leq k \leq 5$, and $A_6$ is $k$-homogeneous on $\Omega$. If $k = 1$, then $A_{m-1} \trianglelefteq S_n \leq S_{m-1}$ and it is easy to deduce $n = m - 1$. Therefore $A_{n+1} = A_6 S_n$ and so, $S_n \leq A_{n+1}$ from which it follows that $n = 1$ which is not the case. If $k \geq 2$ then by [7] $A_6$ can only be $k$-transitive for $k = 2, 3$, or 4. If $k = 2$, then $m = 6$ or 10. Since we have assumed that $m \geq 7$, therefore, if $m = 10$, then $A_{10} = A_6 S_n$ and from $A_8 \trianglelefteq S_n \leq S_8 \times S_2$ we obtain $n \geq 8$ and the order consideration in $A_{10} = A_6 S_n$ leads to $A_{10} = A_6 S_8$. If $k = 3$ or 4, then $m = 6$ and again $A_6 = A_6 S_n$, a contradiction. Since in [9, Theorem D] the role of $S_n$ and $A_6$ may be interchanged, hence we may assume that $A_{m-k} \trianglelefteq A_6 \leq S_{m-k} \times S_k$ and $S_n$ is $k$-homogeneous for some $1 \leq k \leq 5$. However, a contradiction is obtained in this case again.

**Case (ii).** $m = 6, 8$, or 10. If $m = 6$ then $A_6 = A_6 S_n$, a contradiction. If $m = 8$, then $A_8 = A_6 S_n$ from which it follows that $n \geq 7$, but it is easy to see that $A_8$ has no subgroup isomorphic to $S_7$. If $m = 10$, then $A_{10} = A_6 S_n$ from which it follows that $n = 7$ or 8.

Now to rule out the case $n = 7$. We will use [16, Result 1.4]. Using the notation used in [16] the decomposition $A_{10} = A_6 S_7$ is exact and we have $p = 7$ and $|\Delta| = k = 3$ and therefore $A_6$ must be 3-homogeneous which is impossible by [7] unless $A_6$ acts on $\Omega$, $|\Omega| = 6$ in a natural way and this is also a contradiction. However, if $n = 8$, then $A_{10} = A_6 S_8$ and this possibility holds because by Lemma 2.1 we have $A_{10} = A_6 A_8$ and since $A_{10}$ has a subgroup isomorphic to $S_8$, namely $A_8 \langle (1\ 2)(9\ 10) \rangle$ we obtain $A_{10} = A_6 S_8$. $\square$

In this paper we also use the following result which can be proved using the subgroup structure of $L_2(q)$ given in [6].

TABLE 2.1. Primitive groups of degree $k \geq 5$, $k \mid 360$.

| $k$ | Groups |
|---|---|
| 5 | $A_5$, $S_5$ |
| 6 | $A_6$, $S_6$, $A_5$ |
| 8 | $A_8$, $S_8$, $L_3(2)$, $L_3(2).2$, $GA_3(2)$ |
| 9 | $A_9$, $S_9$, $L_2(8)$, $L_2(8).3$, $GA_2(3)$ |
| 10 | $A_{10}$, $S_{10}$, $A_5$, $S_5$, $A_6$, $S_6$, $M_{10}$, $PGL_2(9)$, $P\Gamma L_2(9)$ |
| 12 | $A_{12}$, $S_{12}$, $M_{11}$, $M_{12}$, $L_2(11)$, $L_2(11).2$ |
| 15 | $A_{15}$, $S_{15}$, $A_6$, $S_6$, $A_7$, $A_8$ |
| 18 | $A_{18}$, $S_{18}$, $L_2(17)$, $L_2(17).2$ |
| 20 | $A_{20}$, $S_{20}$, $L_2(19)$, $L_2(19).2$ |
| 24 | $A_{24}$, $S_{24}$, $M_{24}$, $L_2(23)$, $L_2(23).2$ |
| 30 | $A_{30}$, $S_{30}$, $L_2(29)$, $L_2(29).2$ |
| 36 | $A_{36}$, $S_{36}$, $A_9$, $S_9$, $M_{10}$, $PGL_2(9)$, $P\Gamma L_2(9)$, $L_2(8)$, $L_2(8).3$, $U_3(3)$, $U_3(3).2$, $U_4(2)$, $U_4(2).2$, $S_6(2)$, $A_5 \times A_5$, $A_6 \times A_6$ |
| 40 | $A_{40}$, $S_{40}$, $L_4(3)$, $PGL_4(3)$, $U_4(2)$, $U_4(2).2$ |
| 45 | $A_{45}$, $S_{45}$, $M_{10}$, $PGL_2(9)$, $P\Gamma L_2(9)$, $A_{10}$, $S_{10}$, $U_4(2)$, $U_4(2).2$ |
| 60 | $A_{60}$, $S_{60}$, $L_2(59)$, $L_2(59).2$, $A_5 \times A_5$ |
| 72 | $A_{72}$, $S_{72}$, $L_2(71)$, $L_2(71).2$ |
| 90 | $A_{90}$, $S_{90}$, $L_2(89)$, $L_2(89).2$ |
| 120 | $A_{120}$, $S_{120}$, $S_7$, $S_8$, $A_9$, $A_{10}$, $S_{10}$, $L_2(16)$, $L_2(16).2$, $P\Gamma L_2(16)$, $L_3(4)$, $L_3(4).2_1$ $L_3(4).2_2$, $L_3(4).2_3$, $L_3(4).2^2$, $S_4(4)$, $S_4(4).2$, $S_6(2)$, $S_8(2)$, $O_8^+(2)$, $O_8^+(2).2$ |
| 180 | $A_{180}$, $S_{180}$, $L_2(179)$, $L_2(179).2$ |
| 360 | $A_{360}$, $S_{360}$, $L_2(359)$, $L_2(359).2$, $A_6 \times A_6$ |

**LEMMA 2.3.** *It is not possible to decompose the group $L_2(q)$ as the product of $A_6$ and $S_n$, where $n > 4$.*

**PROOF.** By [6, page 213] if $G = L_2(q)$, $q = p^f$, has a subgroup isomorphic to $A_6$, then this subgroup must be of the form $L_2(p^m)$ where $m \mid f$. But $L_2(p^m) \cong A_6$ if and only if $p = 3$ and $m = 2$, hence $G = L_2(3^{2k})$, $k \geq 1$. But again by [6] a symmetric group $S_n$ can be a subgroup of $G$ if and only if $n \leq 4$, a contradiction. □

[15, Lemma 3] is essential in this paper and so we will reproduce it here. We mention that it is not necessary to assume that $B$ is a complete group and our rephrasing of the lemma is as follows.

**LEMMA 2.4.** *Suppose $G = AB$ is such that $A$ is a simple group and $B$ has a unique proper normal subgroup $N$ which is simple. Let $G \ncong A \times B$ and let $M$ be a minimal normal subgroup of $G$. Then one of the following holds:*

(i) *$G = AB = M$ is a simple group*
(ii) *$G = MB$, $M = A \times N$, $N \cong A$*
(iii) *$G = MB$, $M \cong NA$ is simple*
(iv) *$M = A$ or $N$, $[G : AN] = [B : N]$, $AN \cong A \times N$*
(v) *$M \cap X = 1$, $|M| \mid [X : A \cap B]$ for $X \in \{A, B\}$, $|M||A \cap B = |AM/M \cap BM/M|$.*

Our work also depends on the primitive groups of certain degrees. Primitive groups of degree up to 20 were obtained in [11] and up to 1000 in [2]. In Table 2.1, the list

of all primitive groups of degree $k \geq 5$ where $k$ is a divisor of $|A_6| = 360$ is given. Notation for the names of groups in Table 2.1 is taken from [1].

**3. Main results.** In this section, using Lemma 2.4, we characterize finite groups $G = AB$ where $A \cong A_6$ and $B \cong S_n$, $n \geq 5$. But first we deal with the possibilities which arise as different cases in Lemma 2.4.

**Lemma 3.1.** *There is no simple group $M$ such that $M = AB$ where $A \cong A_6$ and $B \cong S_n$, $n \geq 5$, unless $M \cong A_{10}$ and $n = 8$.*

**Proof.** We will assume that $M$ is a simple group having subgroups $A \cong A_6$ and $B \cong S_n$, $n \geq 5$, such that $M = AB$ and derive a contradiction. If $C$ is a maximal subgroup of $G$ containing $B$, then $M = AC$ and $k = [M : C] = [A : A \cap C] \mid 360$. Therefore $M$ is a primitive simple group of degree $k$, where $k$ is a divisor of 360 and $k \geq 5$. By Lemmas 2.2 and 2.3, we know that $M$ cannot be isomorphic to an alternating group or a linear group $L_2(g)$, unless $M \cong A_{10}$ for which the decomposition $A_{10} = A_6S_8$ is possible by Lemma 2.2.

Therefore by Table 2.1 we have the following possibilities for $M : M_{11}$, $M_{12}$, $M_{24}$, $U_3(3)$, $U_4(2)$, $L_4(3)$, $L_3(4)$, $S_4(4)$, $S_6(2)$, $S_8(2)$, $O_8^+(2)$. Since $5 \nmid |U_3(3)|$, therefore $M = U_3(3)$ is impossible. If $M = M_{11}$ or $M_{12}$ then $11 \mid |M|$ and hence $n \geq 11$ which implies that $7 \mid |M|$ a contradiction. If $M = M_{24}$, then $23 \mid |M|$ and so $n \geq 23$ implying that $17 \mid |M|$, a contradiction. The same reasoning rules out $M = L_4(3), S_4(4)$ and $S_8(2)$ considering $13 \mid |M|$ in the first case and $17 \mid |M|$ in the remaining two cases. If $M = U_4(2)$, then as $|U_4(2)| = 2^6 \cdot 3^4 \cdot 5 = |A_6S_n|$ we must have $n = 6$ and therefore $U_4(2) = A_6S_6$, but by [1] $U_4(2)$ has only one conjugacy class of subgroups isomorphic to $S_6$ and hence by [9, Proposition C, page 31] there is $g \in U_4(2)$ such that $U_4(2) = S_6^g S_6$ which by [12, page 26] is impossible. If $M = L_3(4) = A_6S_n$, then $n \geq 7$ but by [1] the group $L_3(4)$ has no subgroup isomorphic to $S_7$. If $M = S_6(2) = A_6S_n$, then as $|S_6(2)| = 2^9 \cdot 3^4 \cdot 5 \cdot 7$ we obtain $7 \leq n \leq 10$ and since by [1] the group $S_6(2)$ has no subgroup isomorphic to $S_9$ hence $n = 7$ or 8. But order consideration yields $n = 8$ and so $S_6(2) = A_6S_8$. By [1], the group $A_6$ cannot be contained in a maximal subgroup of the form $2^5 : S_6$. Again by [1], the group $S_6(2)$ has only one conjugacy class of subgroups isomorphic to $S_8$ and so $S_6(2) = S_8^g S_8$ for some $g \in S_6(2)$ which is impossible by [12, page 26]. Finally, if $M = A_6S_n = O_8^+(2)$, then by [1] $n \leq 8$ and order consideration gives a contradiction. □

**Lemma 3.2.** *Let $G$ be a group such that $G = AB$ where $A \cong A_6$ and $B \cong A_n$, $n \geq 5$, then either $G \cong A \times B$ or one of the following cases holds:*

(i) $G = A_{n+1}$, $n = 5, 9, 14, 19, 29, 35, 39, 44, 59, 71, 89, 119, 179$, *or* $359$

(ii) $G = A_n$, $n \geq 6$, *or*

(iii) $G = A_{10}$, $n = 8$.

**Proof.** First suppose that $G$ is simple. By Lemma 2.1 the cases (i) and (iii) are possible and the case (ii) arise from the trivial factorization of $A_n$. Now assume that the simple group $G$ has the desired decomposition $G = A_6A_n$, $n \geq 5$ and let $C$ be a maximal subgroup of $G$ containing $A_n$. Therefore $G = A_6C$ and $m = [G : C] = [A_6 : A_6 \cap C] \mid 360$.

Maximality of $C$ in $G$ implies that $G$ is a simple primitive permutation group of degree $m$ where $m$ is a divisor of 360. Now by Table 2.1 we know that simple primitive permutation groups are alternating groups, sporadic simple groups and simple groups of Lie type with small orders. We consider the following cases:

(a) The group $G$ is isomorphic to an alternating group. In this case by Lemma 2.1 and what follows after that we obtain all the cases (i), (ii), and (iii) of the lemma.

(b) The group $G \cong L_2(q)$ is a 2-dimensional linear group over the finite field $GF(q)$. In this case by [3] factorization $L_2(q) = A_6A_n$ is possible if and only if $n = 6$ and $q = 9$ which gives the trivial factorization.

(c) The group $G$ is isomorphic to a sporadic simple group. In this case by Table 2.1 we have the following possibilities for $G = M_{11}, M_{12}, M_{24}$. But by [4] the factorization $G = A_6S_n$, $n \geq 5$, is not possible for $G$.

(d) The group $G$ is isomorphic to one of the following linear groups:

$U_3(3)$, $U_4(2)$, $L_4(3)$, $L_3(4)$, $S_4(4)$, $S_6(2)$, $S_8(2)$, $O_8^+(2)$. Since $5 \nmid |U_3(3)|$ therefore $G \neq U_3(3)$. If $G = S_4(4)$ or $S_8(2)$, then since $17 \mid |G|$ we must have $n \geq 17$ and since $13 \nmid |G|$ we get a contradiction. If $G = L_4(3)$, then $13 \mid |G|$ and so $n \geq 13$ which is impossible because $11 \nmid |G|$. If $G = U_4(2) = A_6A_n$, then order consideration yields $n = 6$. But by [1] the maximal subgroup of $U_4(2)$ containing one of the $A_6$ subgroups is conjugate to an $S_6$ subgroup which is maximal in $G$. Therefore $U_4(2) = A_6S_6$ which is impossible by the proof of Lemma 3.1. If $G = L_3(4) = A_6A_n$, then by [1] $n = 6$, a contradiction because $7 \mid |G|$. If $G = S_6(2) = A_6A_n$ then since $3^4 \mid |G|$ and $S_6(2)$ has no subgroup isomorphic to $A_9$ we must have $n = 8$. But $G = A_6A_8$ and $A_8$ is contained in a maximal subgroup of $S_6(2)$ isomorphic to $S_8$ and so $G = A_6S_8$ which is impossible by Lemma 3.1. Finally if $G = O_8^+(2) = A_6A_n$, then by [1] $n \leq 9$ and order consideration gives a contradiction.

Now suppose that $G$ is not isomorphic to $A \times B$ and let $1 \neq M$ be a minimal normal subgroup of $G$. By [14, Lemma 1] $M$ is elementary abelian, $M \cap A = M \cap B = 1$, and $|M|$ divides 360 the order of $A_6$. Thus $M$ is an elementary abelian subgroup of order 2, $2^2$, $2^3$, 3, $3^2$, or 5. By induction, as $G/M = (AM/M)(BM/M)$ with $AM/M \cong A$ and $BM/M \cong B$, that $G/M$ is simple. Hence, either $C_G^{(M)} = M$ or $M \leq Z(G)$. Now $C_G^{(M)} = M$ implies that $A_6 \leq \mathrm{Aut}(M)$, contrary to the possibilities for $M$. Now $M = Z(G)$ and $G/M$ is an alternating group. It follows that $G$ is a covering group of an alternating group, contrary to [14, Theorem 10]. □

**Theorem 3.3.** *Let $G$ be a group such that $G = AB$, $A \cong A_6$ and $B \cong S_n$, $n \geq 5$. Then one of the following cases occurs:*

(a) $G \cong A_6 \times S_n$

(b) $G \cong A_{10} \cong A_6S_8$, $n = 8$

(c) $G \cong (A_6 \times A_6)\langle\tau\rangle$, *$\tau$ an automorphism of order 2 and $A_6 \times A_6$ is the minimal normal subgroup of $G$*, $n = 6$

(d) $G \cong S_{n+1}$, $n = 5, 9, 14, 19, 29, 35, 39, 44, 59, 71, 89, 119, 179, 359$

(e) $G \cong S_n$, $n \geq 6$

(f) $G \cong A_{10} \times Z_2$, $n = 8$

(g) $G \cong (A_6 \times A_n)\langle\tau\rangle$, $n \geq 5$, *where $\tau$ acts as an automorphism of order 2 on both factors.*

**Proof.** Our proof is based on the results of Lemma 2.4 and here we use the same notation used in this lemma. Therefore, let $M$ be a minimal normal subgroup of $G$ and note that $N \cong A_n$. If $G \ncong A \times B$, then one of the following possibilities occurs:

(1) $M = G = AB$ is a simple group. In this case by Lemma 3.1 we have $M \cong A_{10}$ and $n = 8$ and case (b) occurs.

(2) $G = MB$, $M \cong A_6 \times N$, $N \cong A_6$.

In this case $n = 6$ and $G \cong A_6 S_6$, $S_6$ acts on $A_6$ by conjugation and $A_6 \times A_6$ is the minimal normal subgroup of $G$ and this is case (c) in the theorem.

(3) $G = MB$, $M \cong A_6 A_n$ is simple. In this case by Lemma 3.2 three cases occur. If $M = A_{n+1}$, $n = 5, 9, 14, 19, 29, 35, 39, 44, 59, 71, 89, 119, 179$, or $359$, then the same reasoning used in the proof of [15, Theorem 4] yields case (d). If $M = A_n$, then $G = S_n$, $n \geq 6$ and this is the case (e). If $M = A_{10}$ and $n = 8$, then a simple argument forces $G \cong S_{10}$ or $A_{10} \times Z_2$. If $G \cong S_{10}$ we have case (e) again. If we consider the alternating group $A_{10}$ on the set $\{1, 2, \ldots, 10\}$. Then since $A_6$ has a 2-transitive action on 10 letters we obtain $A_{10} = A_6 A_8$ where $A_8$ is the pointwise stabilizer of the set $\{9, 10\}$. Now the set stabilizer of $\{9, 10\}$ is isomorphic to $S_8$ and is a subgroup of $A_{10}$ containing this $A_8$. Therefore $A_{10}\langle(9\ 10)\rangle = A_6 A_8 \langle(9\ 10)\rangle$ implying $A_{10} \times Z_2 \cong A_6 S_8$ which is the case (f).

(4) $M = A_6$ or $A_n$, $[G : A_6 A_n] = 2$, $A_6 A_n \cong A_6 \times A_n$. In this case $G \cong (A_6 \times A_n) \cdot \langle \tau \rangle$ where $\tau$ acts as an outer automorphism of order 2 on both factors and this is the case (g).

(5) $M \cap A = 1$, $M \cap B = 1$ and $|M|$ divides $|A_6|$.

Since $M$ is isomorphic to a direct product of simple groups either $M$ is isomorphic to $A_6$, $A_5$ or $M$ is elementary abelian of order $2, 2^2, 2^3, 3, 3^2$, or $5$. If $M \cong A_6$, then as $MS_n \leq G$ and $M \cap B = 1$, $G = MB \cong A_6 S_n$ with $A_6$ as a minimal normal subgroup. This is the case (4) treated above. Consider $C_G^{(M)}$. Suppose that $A \cap C_G^{(M)} = 1$. Then $A$ is isomorphic to a subgroup of $\mathrm{Aut}(M)$. Considering the possibilities for $M$, this is impossible. Thus, $A \leq C_G^{(M)}$ and by the modular law $C_G^{(M)} = A(B \cap C_G^{(M)})$. Now since $B \cap C_G^{(M)}$ is a normal subgroup of $B$, we must have either $B \cap C_G^{(M)} = 1$, $B$, or $B \cap C_G^{(M)} \cong A_n$. If $B \cap C_G^{(M)} = 1$, then as before $B$ is isomorphic to a subgroup of $\mathrm{Aut}(M)$, contrary to the possibilities for $M$ unless $M \cong A_5$ and $n = 5$. Now $AM$ has index 2 in $G$, so $AM = C_G^{(M)} \times M$ is a normal subgroup of $G$. This is case (4), above. If $B \cap C_G^{(M)} \cong A_n$, then $C_G^{(M)}$ is as in Lemma 3.2. However, none of these groups has a nontrivial center, a contradiction. Thus, we must have $B \leq C_G^{(M)}$ and $M \leq Z(G)$ and hence, $M$ has prime order. By induction, $G/M = (AM/M)(BM/M)$ must be in the list, but $(AM/M) \cap (BM/M) \neq 1$ so only the parts (b), (d), (e), and (f) are possible. If part (e) holds, then we would have $G = BM = B \times M$ contrary to the fact that $A$ has no subgroup of prime index. If part (b) or (d) holds, then $G$ is the covering group of the symmetric group. Now we can see that $BM/M$ must contain an involution which is the product of 2-cycles. It is known, see [10], that such an involution must lift to an element of order 4 in $G$, contrary to the fact that $M \cap B = 1$. (Note that $BM/M$ lifts to $BM$ in $G$, see the argument in [14].) Now suppose that $G/M = A_{10} \times Z_2$ and $n = 8$. Thus $G$ has a normal subgroup of order $2|M|$ which arguing as above must be the center of $G$. It follows that $G$ is a covering group of $A_{10}$. But as the Schur multiplier of $A_{10}$ has order 2 this is impossible. This completes the proof. $\square$

**Acknowledgements.** We would like to thank Professor Gary L. Walls for his interest and guidance in preparation of this manuscript. We also wish to thank the referee for his comments concerning improvements in proving Lemma 3.2 and Theorem 3.3.

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