STATIONARY POINTS FOR SET-VALUED MAPPINGS ON TWO METRIC SPACES

ZEQING LIU, QINGTAO LIU, and SHIN MIN KANG

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ABSTRACT. We give stationary point theorems of set-valued mappings in complete and compact metric spaces. The results in this note generalize a few results due to Fisher.

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1. Introduction and preliminaries. In [2, 4], Fisher and Popa proved fixed point theorems for single-valued mappings on two metric spaces. The purpose of this note is to generalize these results from single-valued mappings into set-valued mappings. In this note, we show stationary point results of set-valued mappings in complete and compact metric spaces.

Let (X, d) and (Y, ρ) be complete metric spaces and $B(X)$ and $B(Y)$ be two families of all nonempty bounded subsets of X and Y , respectively. The function $\delta(A, B)$ with A and B in $B(X)$ is defined as follows:

$$\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\}. \quad (1.1)$$

Define $\delta(A) = \delta(A, A)$. Similarly, the function $\delta'(C, D)$ with C and D in $B(Y)$ is defined as follows:

$$\delta'(C, D) = \sup \{\rho(c, d) : c \in C, d \in D\}. \quad (1.2)$$

A sequence of sets in $B(X)$, $\{A_n : n = 1, 2, \dots\}$ converges to the set A in $B(X)$ if

- (i) each point a in A is the limit of some convergent sequence $\{a_n \in A_n : n = 1, 2, \dots\}$;
- (ii) for arbitrary $\epsilon > 0$, there exists an integer N such that $A_n \subset A_\epsilon$, for $n > N$, where A_ϵ is the union of all open spheres with centers in A and radius ϵ .

Let T be a set-valued mapping of X into $B(X)$. z is a *stationary point* of T if $Tz = \{z\}$. T is *continuous* at x in X if whenever $\{x_n\}$ is a sequence of points in X converging to x , the sequence $\{Tx_n\}$ in $B(X)$ converges to Tx in $B(X)$. If T is continuous at each point x in X , then T is a *continuous mapping* of X into $B(X)$.

The following Lemmas 1.1 and 1.2 were proved in [1, 3], respectively.

LEMMA 1.1. *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space (X, d) which converge to the bounded subsets A and B , respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

LEMMA 1.2. *Let $\{A_n\}$ be a sequence of nonempty subsets of X and let x be a point of X such that $\lim_{n \rightarrow \infty} \delta(A_n, x) = 0$. Then the sequence $\{A_n\}$ converges to the set $\{x\}$.*

2. Stationary point results. Now we prove the following theorem for set-valued mappings.

THEOREM 2.1. *Let (X, d) and (Y, ρ) be complete metric spaces. If T is a continuous mapping of X into $B(Y)$ and S is a continuous mapping of Y into $B(X)$ satisfying the inequalities*

$$\delta(STx, STy) \leq c \max \{ \delta(x, y), \delta(x, STx), \delta(y, STy), \delta'(Tx, Ty) \}, \quad (2.1)$$

$$\delta'(TSx', TSy') \leq c \max \{ \delta'(x', y'), \delta'(x', TSx'), \delta'(y', TSy'), \delta(Sx', Sy') \}, \quad (2.2)$$

for all x, y in X and x', y' in Y , where $0 \leq c < 1$, then ST has a stationary point z in X and TS has a stationary point w in Y . Further $Tz = \{w\}$ and $Sw = \{z\}$.

PROOF. From (2.1) and (2.2), it is easy to see that

$$\begin{aligned} \delta(STA, STB) &\leq c \max \{ \delta(A, B), \delta(A, STA), \delta(B, STB), \delta'(TA, TB) \}, \\ \delta'(TSA', TSB') &\leq c \max \{ \delta'(A', B'), \delta'(A', TSA'), \delta'(B', TSB'), \delta(SA', SB') \}, \end{aligned} \quad (2.3)$$

for all A, B in $B(X)$ and A', B' in $B(Y)$.

Let x be an arbitrary point in X . Define sequences $\{x_n\}$ and $\{y_n\}$ in $B(X)$ and $B(Y)$, respectively, by choosing a point x_n in $(ST)^n x = X_n$ and a point y_n in $T(ST)^{n-1} x = Y_n$ for $n = 1, 2, \dots$. From (2.3) we have

$$\begin{aligned} \delta(X_n, X_{n+1}) &= \delta(STX_{n-1}, STX_n) \\ &\leq c \max \{ \delta(X_{n-1}, X_n), \delta(X_{n-1}, X_n), \delta(X_n, X_{n+1}), \delta'(Y_n, Y_{n+1}) \} \\ &\leq c \max \{ \delta(X_{n-1}, X_n), \delta'(Y_n, Y_{n+1}) \}. \end{aligned} \quad (2.4)$$

Similarly,

$$\delta'(Y_n, Y_{n+1}) \leq c \max \{ \delta'(Y_{n-1}, Y_n), \delta(X_{n-1}, X_n) \}. \quad (2.5)$$

Put $M = \max \{ \delta(x, X_1), \delta'(Y_1, Y_2) \}$. From the above inequalities, we obtain immediately

$$\delta(X_n, X_{n+1}) \leq c^n M, \quad \delta'(Y_n, Y_{n+1}) \leq c^n M, \quad (2.6)$$

for $n \geq 1$. It follows from (2.2) that

$$\begin{aligned} \delta(X_n, X_{n+r}) &\leq \delta(X_n, X_{n+1}) + \dots + \delta(X_{n+r-1}, X_{n+r}) \\ &\leq (c^n + \dots + c^{n+r-1})M \leq \frac{c^n}{1-c}M. \end{aligned} \quad (2.7)$$

Since $c < 1$, then $\delta(X_n, X_{n+r}) \rightarrow 0$ as $n \rightarrow \infty$. So

$$d(x_n, x_{n+r}) \leq \delta(X_n, X_{n+r}) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.8)$$

Thus $\{x_n\}$ is a Cauchy sequence. Completeness of X implies that there exists z in X such that $x_n \rightarrow z$ as $n \rightarrow \infty$. It follows that

$$\begin{aligned} \delta(z, X_n) &\leq \delta(z, x_n) + \delta(x_n, X_n) \\ &\leq \delta(z, x_n) + \delta(X_n, X_n) \\ &\leq \delta(z, x_n) + 2\delta(X_n, X_{n+1}), \end{aligned} \quad (2.9)$$

which implies that $\delta(z, X_n) \rightarrow 0$ as $n \rightarrow \infty$. Similarly, there exists w in Y such that $y_n \rightarrow w$ and $\delta'(w, Y_n) \rightarrow 0$ as $n \rightarrow \infty$. Then

$$\delta'(w, TX_n) \leq \delta'(w, TX_n) = \delta'(w, Y_{n+1}). \quad (2.10)$$

By the continuity of T and [Lemma 1.1](#), we have $\delta'(w, Tz) \rightarrow 0$ as $n \rightarrow \infty$. From [Lemma 1.2](#) it follows that $Tz = \{w\}$. Note that

$$\begin{aligned} \delta(STz, x_n) &\leq \delta(STz, X_n) \\ &\leq c \max \{ \delta(z, X_{n-1}), \delta(z, STz), \delta(X_{n-1}, X_n), \delta'(Tz, TX_{n-1}) \}. \end{aligned} \quad (2.11)$$

Letting n tend to infinity, we have

$$\delta(STz, z) \leq c \max \{ \delta(STz, z), 0 \}, \quad (2.12)$$

which implies that $STz = \{z\} = Sw$. Similarly, we can show that w is a stationary point of TS . This completes the proof of the theorem. $\square$

THEOREM 2.2. *Let (X, d) be a complete metric space, and let S and T be continuous mappings of X into $B(X)$ and map bounded set into bounded set. If S and T satisfy the inequalities*

$$\begin{aligned} \delta(STx, STy) &\leq c \max \{ \delta(x, y), \delta(x, STx), \delta(y, STy), \\ &\quad \delta(x, STy), \delta(y, STx), \delta(Tx, Ty) \}, \end{aligned} \quad (2.13)$$

$$\begin{aligned} \delta(TSx, TSy) &\leq c \max \{ \delta(x, y), \delta(x, TSx), \delta(y, TSy), \\ &\quad \delta(x, TSy), \delta(y, TSx), \delta(Sx, Sy) \}, \end{aligned} \quad (2.14)$$

for all x, y in X , where $0 \leq c < 1$, then ST has a stationary point z and TS has a stationary point w . Further $Tz = \{w\}$ and $Sw = \{z\}$. If $z = w$, then z is the unique common stationary point of S and T .

PROOF. Let x be an arbitrary point in X . Define a sequence of sets $\{X_n\}$ by $T(ST)^{n-1}x = X_{2n-1}$, $(ST)^n x = X_{2n}$ for $n \geq 1$ and $X_0 = \{x\}$.

Now suppose that $\{\delta(X_n)\}$ is unbounded. Then the real-valued sequence $\{a_n\}$ is unbounded, where $a_{2n-1} = \delta(X_{2n-1}, X_3)$, $a_{2n} = \delta(X_{2n}, X_2)$ for $n \geq 1$ and so there exists an integer k such that

$$a_k > \frac{c}{1-c} \max \{ \delta(x, X_2), \delta(X_1, X_3) \}, \quad (2.15)$$

$$a_k > \max \{ a_1, \dots, a_{k-1} \}. \quad (2.16)$$

Suppose that k is even. Put $k = 2n$. From (2.15) and (2.16) we have

$$\begin{aligned} c\delta(X_{2r}, x) &\leq c[\delta(X_{2r}, X_2) + \delta(X_2, x)] < \delta(X_{2n}, X_2), \\ c\delta(X_{2r-1}, X_1) &\leq c[\delta(X_{2r-1}, X_3) + \delta(X_3, X_1)] < \delta(X_{2n}, X_2). \end{aligned} \quad (2.17)$$

That is,

$$\delta(X_{2n}, X_2) > c \max \{ \delta(X_{2r}, x), \delta(X_{2r-1}, X_1) : 1 \leq r \leq n \}. \quad (2.18)$$

We now prove that the following inequality is true for $m \geq 1$:

$$\delta(X_{2n}, X_2) \leq c^m \max \{ \delta(X_{2r}, X_{2s}), \delta(X_{2r'-1}, X_{2s'-1}) : 1 \leq r, s \leq n, 2 \leq r', s' \leq n \}. \quad (2.19)$$

From (2.13) we have

$$\begin{aligned} \delta(X_{2n}, X_2) &= \delta(STX_{2n-2}, STx) \\ &\leq c \max \{ \delta(X_{2n-2}, x), \delta(X_{2n-2}, X_{2n}), \delta(x, X_2), \\ &\quad \delta(x, X_{2n}), \delta(X_{2n-2}, X_2), \delta(X_{2n-1}, X_1) \}. \end{aligned} \quad (2.20)$$

It follows from (2.16) and (2.18) that

$$\delta(X_{2n}, X_2) \leq c \delta(X_{2n-2}, X_{2n}). \quad (2.21)$$

Now suppose that (2.19) is true for some m . From (2.13), (2.14), (2.16), and (2.18) we have

$$\begin{aligned} \delta(X_{2n}, X_2) &\leq c^m \max \{ \delta(X_{2r}, X_{2s}), \delta(X_{2r'-1}, X_{2s'-1}) : 1 \leq r, s \leq n, 2 \leq r', s' \leq n \} \\ &\leq c^{m+1} \max \{ \delta(X_{2r-2}, X_{2s-2}), \delta(X_{2r-2}, X_{2r}), \delta(X_{2s-2}, X_{2s}), \\ &\quad \delta(X_{2r-2}, X_{2s}), \delta(X_{2s-2}, X_{2r}), \delta(X_{2r-1}, X_{2s-1}), \\ &\quad \delta(X_{2r'-3}, X_{2s'-3}), \delta(X_{2r'-3}, X_{2r'-1}), \\ &\quad \delta(X_{2s'-3}, X_{2s'-1}) : 1 \leq r, s \leq n, 2 \leq r', s' \leq n \} \\ &\leq c^{m+1} \max \{ \delta(X_{2r}, X_{2s}), \delta(X_{2r'-1}, X_{2s'-1}) : 1 \leq r, s \leq n, 2 \leq r', s' \leq n \}. \end{aligned} \quad (2.22)$$

So (2.19) is true for all $m \geq 1$. Letting m tend to infinity, from (2.16) and (2.18) we have $0 < \delta(X_{2n}, X_2) \leq 0$, which is impossible. Similarly, when k is odd, $2n-1$, say, we also have $0 < \delta(X_{2n-1}, X_3) \leq 0$, which is also impossible. Hence $\{\delta(X_n)\}$ is bounded.

Let $M = \sup \{ \delta(X_r, X_s) : r, s = 0, 1, 2, \dots \} < \infty$. For arbitrary $\epsilon > 0$, choose a positive integer N such that $c^N M < \epsilon$. Thus for m, n greater than $2N$ with m and n both even or both odd, from (2.13) and (2.14) we have

$$\begin{aligned} \delta(X_m, X_n) &\leq c \max \{ \delta(X_{m-2}, X_{n-2}), \delta(X_{m-2}, X_m), \delta(X_{n-2}, X_n), \\ &\quad \delta(X_{m-2}, X_n), \delta(X_{n-2}, X_m), \delta(X_{m-1}, X_{n-1}) \} \\ &\leq c \max \{ \delta(X_r, X_s), \delta(X_r, X_{r'}), \delta(X_s, X_{s'}) : \\ &\quad m-2 \leq r, r' \leq m, n-2 \leq s, s' \leq n \} \\ &\leq c^N \max \{ \delta(X_r, X_s), \delta(X_r, X_{r'}), \delta(X_s, X_{s'}) : \\ &\quad m-2N \leq r, r' \leq m, n-2N \leq s, s' \leq n \} \\ &\leq c^N M < \epsilon. \end{aligned} \quad (2.23)$$

So $\delta(X_{2n})$ and $\delta(X_{2n+1}) \rightarrow 0$ as $n \rightarrow \infty$. Take a point x_n in X_n for $n \geq 1$. Since $d(x_{2n}, x_{2n+2p}) \leq \delta(X_{2n}, X_{2n+2p}) \rightarrow 0$ as $n \rightarrow \infty$, hence $\{x_{2n}\}$ is a Cauchy sequence. Completeness of X implies that $\{x_{2n}\}$ has a limit z in X . Note that

$$\delta(z, X_{2n}) \leq \delta(z, x_{2n}) + \delta(x_{2n}, X_{2n}) \leq \delta(z, x_{2n}) + \delta(X_{2n}). \quad (2.24)$$

That is, $\delta(z, X_{2n}) \rightarrow 0$ as $n \rightarrow \infty$. Similarly $\{x_{2n+1}\}$ converges to some point w in X and $\delta(w, X_{2n+1}) \rightarrow 0$ as $n \rightarrow \infty$. Since $\delta(w, TX_{2n}) = \delta(w, X_{2n+1})$, by the continuity of T and Lemma 1.1, we have $\delta(w, Tz) \rightarrow 0$ as $n \rightarrow \infty$. From Lemma 1.2 it follows that $Tz = \{w\}$. In view of (2.13), we obtain that

$$\begin{aligned} \delta(STz, x_{2n}) &\leq \delta(STz, X_{2n}) \\ &\leq c \max \{ \delta(z, X_{2n-2}), \delta(z, STz), \delta(X_{2n-2}, X_{2n}), \\ &\quad \delta(z, X_{2n}), \delta(X_{2n-2}, STz), \delta(Tz, X_{2n-1}) \}, \end{aligned} \quad (2.25)$$

which implies that

$$\delta(STz, z) \leq c \max \{ \delta(z, STz), 0 \} \quad (2.26)$$

as $n \rightarrow \infty$. Since $c < 1$, $\delta(STz, z) = 0$. Therefore $STz = \{z\} = Sw$ and $TSw = Tz = \{w\}$.

Now suppose that $z = w$ and that z' is the second common stationary point of S and T . Using (2.1)

$$\begin{aligned} \delta(z, z') &= \delta(STz, STz') \\ &\leq c \max \{ \delta(z, z'), \delta(z, STz), \delta(z', STz'), \\ &\quad \delta(z', STz), \delta(z, STz'), \delta(Tz, Tz') \} \\ &\leq c \delta(z, z'). \end{aligned} \quad (2.27)$$

So $z = z'$ and this completes the proof of the theorem. $\square$

REMARK 2.3. If we use single-valued mappings in place of set-valued mappings in Theorems 2.1 and 2.2, Theorems 2 and 3 of Fisher [2] can be attained.

REMARK 2.4. The following example demonstrates that the continuity of S and T in Theorems 2.1 and 2.2 is necessary.

EXAMPLE 2.5. Let $X = \{0\} \cup \{1/n : n \geq 1\} = Y$ with the usual metric. Define mappings S, T by $T0 = \{1\}$, $T(1/n) = \{1/2n\}$ for $n \geq 1$ and $S = T$. It is easy to prove that all the conditions of Theorems 2.1 and 2.2 are satisfied except that the mappings S and T are continuous. But ST and TS have no stationary points.

Now we give the following theorem for the compact metric spaces.

THEOREM 2.6. Let (X, d) and (Y, ρ) be compact metric spaces. If T is a continuous mapping of X into $B(Y)$ and S is a continuous mapping of Y into $B(X)$ satisfying the following inequalities:

$$\delta(STx, STy) < \max \{ \delta(x, y), \delta(x, STx), \delta(y, STy), \delta'(Tx, Ty) \}, \quad (2.28)$$

$$\delta'(TSx', TSy') < \max \{ \delta'(x', y'), \delta'(x', TSx'), \delta'(y', TSy'), \delta(Sx', Sy') \}, \quad (2.29)$$

for all distinct x, y in X and distinct x', y' in Y , then ST has a stationary point z and TS has a stationary point w . Further $Tz = \{w\}$ and $Sw = \{z\}$.

PROOF. Suppose that the right-hand sides of inequalities (2.28) and (2.29) are positive for all distinct x, y in X and distinct x', y' in Y . Define the real-valued function

$f(x, y)$ in $X \times X$ as follows:

$$f(x, y) = \frac{\delta(STx, STy)}{\max\{\delta(x, y), \delta(x, STx), \delta(y, STy), \delta'(Tx, Ty)\}}. \quad (2.30)$$

Since S and T are continuous, f is continuous and achieves the maximum value s on the compact metric space $X \times X$. Inequality (2.28) implies $s < 1$. That is,

$$\delta(STx, STy) \leq s \max\{\delta(x, y), \delta(x, STx), \delta(y, STy), \delta'(Tx, Ty)\} \quad (2.31)$$

for all distinct x, y in X . It is obvious that (2.31) is also true for $x = y$. Similarly, there exists $t < 1$ such that

$$\delta'(TSx', TSy') \leq t \max\{\delta'(x', y'), \delta'(x', TSx'), \delta'(y', TSy'), \delta(Sx', Sy')\} \quad (2.32)$$

for all x', y' in Y . So Theorem 2.6 follows immediately from Theorem 2.1.

Now suppose that there exist z, z' in X such that

$$\max\{\delta(z, z'), \delta(z, STz), \delta(z', STz'), \delta'(Tz, Tz')\} = 0, \quad (2.33)$$

which implies $\{z\} = \{z'\} = STz$ and $Tz = Tz'$, a singleton, $\{w\}$, say. Therefore we have $STz = Sw = \{z\}$, $TSw = Tz = \{w\}$. If there exist w, w' in Y such that

$$\max\{\delta'(w, w'), \delta'(w, TSw), \delta'(w', TSw'), \delta(Sw, Sw')\} = 0. \quad (2.34)$$

Similarly, we also have $STz = Sw = \{z\}$ and $TSw = Tz = \{w\}$. This completes the proof of the theorem. $\square$

REMARK 2.7. Theorem 4 of Fisher [2] is a particular case of our Theorem 2.6 if the set-valued mappings in Theorem 2.6 are replaced by single-valued mappings.

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ZEQING LIU: DEPARTMENT OF MATHEMATICS, LIAONING NORMAL UNIVERSITY, P.O. BOX 200, DALIAN, LIAONING 116029, CHINA

E-mail address: zeqingliu@sina.com.cn

QINGTAO LIU: DEPARTMENT OF MATHEMATICS, LIAONING NORMAL UNIVERSITY, P.O. BOX 200, DALIAN, LIAONING 116029, CHINA

SHIN MIN KANG: DEPARTMENT OF MATHEMATICS, GYEONGSANG NATIONAL UNIVERSITY, CHINJU 660-701, KOREA

E-mail address: smkang@nongae.gsnu.ac.kr