ON CERTAIN ANALYTIC UNIVALENT FUNCTIONS

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ABSTRACT. We consider the class of analytic functions \( B(\alpha) \) to investigate some properties for this class. The angular estimates of functions in the class \( B(\alpha) \) are obtained. Finally, we derive some interesting conditions for the class of strongly starlike and strongly convex of order \( \alpha \) in the open unit disk.

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1. Introduction. Let \( A \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in the open unit disk \( U = \{ z : |z| < 1 \} \). A function \( f(z) \) belonging to \( A \) is said to be starlike of order \( \alpha \) if it satisfies

\[
\text{Re}\left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in U)
\]

for some \( \alpha (0 \leq \alpha < 1) \). We denote by \( S^*_\alpha \) the subclass of \( A \) consisting of functions which are starlike of order \( \alpha \) in \( U \). Also, a function \( f(z) \) belonging to \( A \) is said to be convex of order \( \alpha \) if it satisfies

\[
\text{Re}\left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U)
\]

for some \( \alpha (0 \leq \alpha < 1) \). We denote by \( C_\alpha \) the subclass of \( A \) consisting of functions which are convex of order \( \alpha \) in \( U \).

If \( f(z) \in A \) satisfies

\[
\left| \arg\left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U)
\]

for some \( \alpha (0 \leq \alpha < 1) \), then \( f(z) \) said to be strongly starlike of order \( \alpha \) in \( U \), and this class denoted by \( \tilde{S}^*_\alpha \).

If \( f(z) \in A \) satisfies

\[
\left| \arg\left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U)
\]

for some \( \alpha (0 \leq \alpha < 1) \), then we say that \( f(z) \) is strongly convex of order \( \alpha \) in \( U \), and we denote by \( \tilde{C}_\alpha \) the class of all such functions.
The object of the present paper is to investigate various properties of the following class of analytic functions defined as follows.

**Definition 1.1.** A function \( f(z) \in \mathbb{A} \) is said to be a member of the class \( B(\alpha) \) if and only if
\[
\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \alpha
\]  
for some \( \alpha (0 \leq \alpha < 1) \) and for all \( z \in U \).

Note that condition (1.6) implies
\[
\text{Re} \left( \frac{z^2 f'(z)}{f^2(z)} \right) > \alpha.
\]  
(1.7)

**2. Main results.** In order to derive our main results, we have to recall here the following lemmas.

**Lemma 2.1** (see [2]). Let \( f(z) \in \mathbb{A} \) satisfy the condition
\[
\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 \quad (z \in U),
\]  
(2.1)
then \( f \) is univalent in \( U \).

**Lemma 2.2** (see [1]). Let \( w(z) \) be analytic in \( U \) and such that \( w(0) = 0 \). Then if \( |w(z)| \) attains its maximum value on circle \( |z| = r < 1 \) at a point \( z_0 \in U \), we have
\[
z_0 w'(z_0) = kw(z_0),
\]  
(2.2)
where \( k \geq 1 \) is a real number.

**Lemma 2.3** (see [3]). Let a function \( p(z) \) be analytic in \( U \), \( p(0) = 1 \), and \( p(z) \neq 0 \) (\( z \in U \)). If there exists a point \( z_0 \in U \) such that
\[
\left| \frac{\arg(p(z))}{\arg(p(z_0))} \right| < \frac{\pi}{2} \alpha, \quad \text{for } |z| < |z_0|, \quad \left| \arg(p(z_0)) \right| = \frac{\pi}{2} \alpha,
\]  
(2.3)
with \( 0 < \alpha \leq 1 \), then we have
\[
\frac{z_0 p'(z_0)}{p(z_0)} = ik \alpha,
\]  
(2.4)
where
\[
k \geq \frac{1}{2} (a + \frac{1}{a}) \geq 1 \quad \text{when } \arg(p(z_0)) = \frac{\pi}{2} \alpha,
\]  
\[
k \leq -\frac{1}{2} (a + \frac{1}{a}) \leq -1 \quad \text{when } \arg(p(z_0)) = -\frac{\pi}{2} \alpha,
\]  
(2.5)
\[
p(z_0)^{1/\alpha} = \pm ai, \quad (a > 0).
\]

We begin with the statement and the proof of the following result.
Theorem 2.4. If \( f(z) \in \mathbb{A} \) satisfies
\[
\left| \frac{(zf(z))'' - 2zf'(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < \frac{1 - \alpha}{2 - \alpha} \quad (z \in U),
\]
for some \( \alpha \) \((0 \leq \alpha < 1)\), then \( f(z) \in B(\alpha) \).

Proof. We define the function \( w(z) \) by
\[
\frac{z^2f'(z)}{f^2(z)} = 1 + (1 - \alpha)w(z).
\]
Then \( w(z) \) is analytic in \( U \) and \( w(0) = 0 \). By the logarithmic differentiations, we get from (2.7) that
\[
\frac{(zf(z))'' - 2zf'(z)}{f'(z)} - \frac{zf'(z)}{f(z)} = \frac{(1 - \alpha)zw'(z)}{1 + (1 - \alpha)w(z)}.
\]
Suppose there exists \( z_0 \in U \) such that
\[
\max_{|z|<|z_0|} |w(z)| = |w(z_0)| = 1,
\]
then from Lemma 2.2, we have (2.2).

Letting \( w(z_0) = e^{i\theta} \), from (2.8), we have
\[
\left| \frac{(z_0f(z_0))'' - 2z_0f'(z_0)}{f'(z_0)} - \frac{z_0f'(z_0)}{f(z_0)} \right| = \left| \frac{(1 - \alpha)ke^{i\theta}}{1 + (1 - \alpha)e^{i\theta}} \right| \geq \frac{1 - \alpha}{2 - \alpha},
\]
which contradicts our assumption (2.6). Therefore \( |w(z)| < 1 \) holds for all \( z \in U \). We finally have
\[
\left| \frac{z^2f'(z)}{f^2(z)} - 1 \right| = (1 - \alpha) |w(z)| < 1 - \alpha \quad (z \in U),
\]
that is, \( f(z) \in B(\alpha) \).

Corollary 2.5. If \( f(z) \in \mathbb{A} \) satisfies
\[
\left| \frac{(zf(z))'' - 2zf'(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < \frac{1}{2} \quad (z \in U),
\]
then \( f(z) \) is univalent in \( U \).

Next, we prove the following theorem.

Theorem 2.6. Let \( f(z) \in \mathbb{A} \). If \( f(z) \in B(\alpha) \), then
\[
\left| \frac{f(z)}{z} \right| < \frac{\pi}{2} \alpha \quad (z \in U),
\]
for some \( \alpha \) \((0 < \alpha < 1)\) and \((2/\pi)\tan^{-1} \alpha - \alpha = 1\).
**Proof.** We define the function $p(z)$ by

$$f(z)z = p(z) = 1 + \sum_{n=2}^{\infty} a_n z^{n-1}. \quad (2.14)$$

Then we see that $p(z)$ is analytic in $U$, $p(0) = 1$, and $p(z) \neq 0 \ (z \in U)$. It follows from (2.14) that

$$\frac{z^2 f'(z)}{f^2(z)} = \frac{1}{p(z)} \left(1 + \frac{zp'(z)}{p(z)}\right). \quad (2.15)$$

Suppose there exists a point $z_0 \in U$ such that

$$\left|\arg(p(z))\right| < \frac{\pi}{2} \alpha, \quad \text{for } |z| < |z_0|, \quad \left|\arg(p(z_0))\right| = \frac{\pi}{2} \alpha. \quad (2.16)$$

Then, applying Lemma 2.3, we can write that

$$z_0 p'(z_0) p(z_0) = i k \alpha, \quad (2.17)$$

where

$$k \geq \frac{1}{2} \left(a + \frac{1}{a}\right) \geq 1 \quad \text{when } \arg(p(z_0)) = \frac{\pi}{2} \alpha,$$

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a}\right) \leq -1 \quad \text{when } \arg(p(z_0)) = -\frac{\pi}{2} \alpha, \quad (2.18)$$

$$p(z_0)^{1/\alpha} = \pm ai, \quad (a > 0).$$

Therefore, if $\arg(p(z_0)) = \pi \alpha / 2$, then

$$\frac{z_0^2 f'(z_0)}{f^2(z_0)} = \frac{1}{p(z_0)} \left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right) = a^{-\alpha} e^{-i \pi \alpha / 2} (1 + ik \alpha). \quad (2.19)$$

This implies that

$$\arg\left(\frac{z_0^2 f'(z_0)}{f^2(z_0)}\right) = \arg\left(\frac{1}{p(z_0)} \left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\right)\right) = -\frac{\pi}{2} \alpha + \arg(1 + i \alpha k) \geq -\frac{\pi}{2} \alpha + \tan^{-1} \alpha$$

$$= \frac{\pi}{2} \left(\frac{2}{\pi} \tan^{-1} \alpha - \alpha\right) = \frac{\pi}{2} \quad (2.20)$$

if

$$\frac{2}{\pi} \tan^{-1} \alpha - \alpha = 1. \quad (2.21)$$

Also, if $\arg(p(z_0)) = -\pi \alpha / 2$, we have

$$\arg\left(\frac{z_0^2 f'(z_0)}{f^2(z_0)}\right) \leq -\frac{\pi}{2} \quad (2.22)$$

if

$$\frac{2}{\pi} \tan^{-1} \alpha - \alpha = 1. \quad (2.23)$$

These contradict the assumption of the theorem.
Thus, the function \( p(z) \) has to satisfy
\[
| \arg(p(z)) | < \frac{\pi}{2} \alpha \quad (z \in U)
\] (2.24)
or
\[
| \arg\left( \frac{f(z)}{z} \right) | < \frac{\pi}{2} \alpha \quad (z \in U). \tag{2.25}
\]

This completes the proof. \( \square \)

Now, we prove the following theorem.

**Theorem 2.7.** Let \( p(z) \) be analytic in \( U \), \( p(z) \neq 0 \) in \( U \) and suppose that
\[
| \arg\left( p(z) + \frac{z^3 f'(z)}{f^2(z)} p'(z) \right) | < \frac{\pi}{2} \alpha \quad (z \in U), \tag{2.26}
\]
where \( 0 < \alpha < 1 \) and \( f(z) \in B(\alpha) \), then we have
\[
| \arg(p(z)) | < \frac{\pi}{2} \alpha \quad (z \in U). \tag{2.27}
\]

**Proof.** Suppose there exists a point \( z_0 \in U \) such that
\[
| \arg(p(z_0)) | < \frac{\pi}{2} \alpha, \quad \text{for} \quad |z| < |z_0|, \quad | \arg(p(z_0)) | = \frac{\pi}{2} \alpha. \tag{2.28}
\]

Then, applying Lemma 2.3, we can write that
\[
\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha, \tag{2.29}
\]
where
\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg(p(z_0)) = \frac{\pi}{2} \alpha,
\]
\[
k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg(p(z_0)) = -\frac{\pi}{2} \alpha, \tag{2.30}
\]
\[
p(z_0)^{1/\alpha} = \pm a i, \quad (a > 0).
\]

Then it follows that
\[
\arg\left( p(z_0) + \frac{z_0^3 f'(z_0)}{f^2(z_0)} p'(z_0) \right) = \arg\left( p(z_0) \left( 1 + \frac{z_0^2 f'(z_0) z p'(z_0)}{f^2(z_0)} \right) \right) = \arg\left( p(z_0) \left( 1 + i \frac{z_0^2 f'(z_0)}{f^2(z_0)} \alpha k \right) \right). \tag{2.31}
\]

When \( \arg(p(z_0)) = \pi \alpha/2 \), we have
\[
\arg\left( p(z_0) + \frac{z_0^3 f'(z_0)}{f^2(z_0)} p'(z_0) \right) = \arg(p(z_0)) + \arg\left( 1 + i \frac{z_0^2 f'(z_0)}{f^2(z_0)} \alpha k \right) > \frac{\pi}{2} \alpha, \tag{2.32}
\]
because
\[
\text{Re} \left( \frac{z_0^2 f'(z_0)}{f^2(z_0)} \alpha k \right) > 0 \quad \text{and therefore} \quad \arg\left( 1 + i \frac{z_0^2 f'(z_0)}{f^2(z_0)} \alpha k \right) > 0. \tag{2.33}
\]
Similarly, if \( \arg(p(z_0)) = -\pi \alpha/2 \), then we obtain that
\[
\arg \left( p(z_0) + \frac{z_0^2 f'(z_0)}{f^2(z_0)} p'(z_0) \right) = \arg(p(z_0)) + \arg \left( 1 + i \frac{z_0^2 f'(z_0)}{f^2(z_0)} \alpha k \right) < -\frac{\pi}{2} \alpha, \tag{2.34}
\]
because
\[
\text{Re} \frac{z_0^2 f'(z_0)}{f^2(z_0)} \alpha k < 0 \quad \text{and therefore} \quad \arg \left( 1 + i \frac{z_0^2 f'(z_0)}{f^2(z_0)} \alpha k \right) < 0. \tag{2.35}
\]
Thus we see that (2.32) and (2.34) contradict our condition (2.26). Consequently, we conclude that
\[
\left| \arg(p(z)) \right| < \frac{\pi}{2} \alpha \quad (z \in U). \tag{2.36}
\]
Taking \( p(z) = z f'(z)/f(z) \) in Theorem 2.7, we have the following corollary.

**Corollary 2.8.** If \( f(z) \in \mathbb{A} \) satisfying
\[
\left| \arg \left( \frac{z f'(z)}{f(z)} + \frac{z^3 f''(z)}{f^3(z)} \left( (z f'(z))' - \frac{z (f'(z))^2}{f(z)} \right) \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U), \tag{2.37}
\]
where \( 0 < \alpha < 1 \) and \( f(z) \in B(\alpha) \), then \( f(z) \in \mathbb{S}^* \).

Taking \( p(z) = 1 + z f''(z)/f'(z) \) in Theorem 2.7, we have the following corollary.

**Corollary 2.9.** If \( f(z) \in \mathbb{A} \) satisfying
\[
\left| \arg \left( \frac{(z f'(z))^2}{f'(z)} + \frac{z^3}{f^3(z)} \left( (z f'(z))' - \frac{z (f'(z))^2}{f(z)} \right) \right) \right| < \frac{\pi}{2} \alpha, \tag{2.38}
\]
where \( 0 < \alpha < 1 \) and \( f(z) \in B(\alpha) \), then \( f(z) \in \mathbb{C}^* \).

**References**


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