ON IDEALS OF IMPLICATIVE SEMIGROUPS

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ABSTRACT. We introduce the notion of ideals in implicative semigroups, and then state the characterizations of the ideals.

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1. Introduction. The notions of implicative semigroup and ordered filter were introduced by Chan and Shum [3]. The first is a generalization of implicative semilattice (see Nemitz [6] and Blyth [2]) and has a close relation with implication in mathematical logic and set theoretic difference (see Birkhoff [1] and Curry [4]). For the general development of implicative semilattice theory the ordered filters play an important role which is shown by Nemitz [6]. Motivated by this, Chan and Shum [3] established some elementary properties, and constructed quotient structure of implicative semigroups via ordered filters. Jun et al. [5] discussed ordered filters of implicative semigroups. In this paper, we introduce the notion of ideals in implicative semigroups. By introducing special subsets of an implicative semigroups, we provide a condition for the special subset to be an ideal. We establish two characterizations of ideals.

2. Preliminaries. We recall some definitions and results. By a negatively partially ordered semigroup (briefly, n.p.o. semigroup) we mean a set $S$ with a partial ordering $\leq$ and a binary operation $\cdot$ such that for all $x, y, z \in S$, we have

\begin{enumerate}
  \item $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
  \item $x \leq y$ implies $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$,
  \item $x \cdot y \leq x$ and $x \cdot y \leq y$.
\end{enumerate}

An n.p.o. semigroup $(S; \leq, \cdot)$ is said to be implicative if there is an additional binary operation $\ast : S \times S \to S$ such that for any elements $x, y, z$ of $S$,

\begin{enumerate}
  \item $z \leq x \ast y$ if and only if $z \cdot x \leq y$.
\end{enumerate}

An implicative semigroup $(S; \leq, \cdot, \ast)$ is said to be commutative if it satisfies

\begin{enumerate}
  \item $x \cdot y = y \cdot x$ for all $x, y \in S$, that is, $(S, \cdot)$ is a commutative semigroup.
\end{enumerate}

In any implicative semigroup $(S; \leq, \cdot, \ast)$, $x \ast x = y \ast y$ for every $x, y \in S$ and this element is the greatest element, written 1, of $(S, \leq)$.

PROPOSITION 2.1 (see [3, Theorem 1.4]). Let $S$ be an implicative semigroup. Then for every $x, y, z \in S$, the following hold:

\begin{enumerate}
  \item $x \leq 1, x \ast x = 1, x = 1 \ast x$,
  \item $x \leq y \ast (x \cdot y)$,
\end{enumerate}
(8) \( x \leq x \ast x^2 \),
(9) \( x \leq y \ast x \),
(10) if \( x \leq y \) then \( x \ast z \geq y \ast z \) and \( z \ast x \leq z \ast y \),
(11) \( x \leq y \) if and only if \( x \ast y = 1 \),
(12) \( x \ast (y \ast z) = (x \cdot y) \ast z \),
(13) if \( S \) is commutative then \( x \ast y \leq (s \cdot x) \ast (s \cdot y) \) for all \( s \) in \( S \).

Now we note important elementary properties of a commutative implicative semigroup, which follows from (5), (6), and (12).

**Observation 2.2.** If \( S \) is a commutative implicative semigroup, then for any \( x, y, z \in S \),

(14) \( x \ast (y \ast z) = y \ast (x \ast z) \),
(15) \( y \ast z \leq (x \ast y) \ast (x \ast z) \),
(16) \( x \leq (x \ast y) \ast y \).

3. Ideals of implicative semigroups. In what follows let \( S \) denote an implicative semigroup unless otherwise specified. We begin by defining the notion of ideals of \( S \).

**Definition 3.1.** A subset \( I \) of \( S \) is called an **ideal** of \( S \) if

(I1) \( x \in S \) and \( a \in I \) imply \( x \ast a \in I \),
(I2) \( x \in S \) and \( a, b \in I \) imply \( (a \ast (b \ast x)) \ast x \in I \).

**Example 3.2.** Consider an implicative semigroup \( S := \{1, a, b, c, d, 0\} \) with Cayley tables (Tables 3.1 and 3.2) and Hasse diagram (Figure 3.1) as follows:

**Table 3.1**

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We know that \( \{1, a, b\} \) is an ideal of \( S \), but \( \{1, a\} \) is not an ideal of \( S \), since \( (a \ast (a \ast b)) \ast b = b \notin \{1, a\} \).
**Lemma 3.3.** Every ideal of $S$ contains 1.

**Proof.** The proof follows from (6) and (I1).

**Lemma 3.4.** If $I$ is an ideal of $S$, then $(a * x) * x \in I$ for all $a \in I$ and $x \in S$.

**Proof.** The proof follows by taking $b = a$ and $a = 1$ in (I2).

**Corollary 3.5.** Let $I$ be an ideal of $S$. If $a \in I$ and $a \leq x$, then $x \in I$.

**Proof.** Let $a \in I$ and $x \in S$ be such that $a \leq x$. Using (6) and Lemma 3.4, we have $x = 1 * x = (a * x) * x \in I$. This completes the proof.

**Lemma 3.6.** Let $I$ be a subset of $S$ such that

(I3) $1 \in I$,

(I4) $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$ for all $x, y, z \in S$.

If $a \in I$ and $a \leq x$, then $x \in I$.

**Proof.** Let $a \in I$ and $x \in S$ be such that $a \leq x$. Then $x * (a * 1) = x * 1 = 1 \in I$ by (6) and (I3), and so $x = x * 1 \in I$ by (I4). This completes the proof.

The following is a characterization of ideals.

**Theorem 3.7.** Let $S$ be a commutative implicative semigroup. A subset $I$ of $S$ is an ideal of $S$ if and only if it satisfies conditions (I3) and (I4).

**Proof.** Let $I$ be an ideal of $S$. Then $1 \in I$ by Lemma 3.3. Let $x, y, z \in S$ be such that $x * (y * z) \in I$ and $y \in I$. Using Lemma 3.4, we get $(y * z) * z \in I$. It follows from (6), (15), and (I2) that

$$x * z = 1 * (x * z) = (((y * z) * z) * ((x * (y * z)) * (x * z))) * (x * z) \in I. \quad (3.1)$$

Conversely, assume that $I$ satisfies conditions (I3) and (I4). Let $x \in S$ and $a \in I$. Since $x * (a * a) = x * 1 = 1 \in I$ by (I3), it follows from (I4) that $x * a \in I$, that is, (I1) holds. Since $(a * x) * (a * x) = 1 \in I$, we have $(a * x) * x \in I$ by (I4). Note from (15) that

$$((a * x) * x) * ((b * (a * x)) * (b * x)) = 1, \quad (3.2)$$

that is,

$$(a * x) * x \leq (b * (a * x)) * (b * x) \quad (3.3)$$

for all $b \in I$. Thus, by Lemma 3.6, we have $(b * (a * x)) * (b * x) \in I$. Using (I4), we conclude that $(b * (a * x)) * x \in I$ which proves (I2). Hence $I$ is an ideal of $S$.  \[\square\]
For any \( u, v \in S \), consider a set

\[
S(u, v) = \{ z \in S \mid u \ast (v \ast z) = 1 \}.
\] (3.4)

In Example 3.2, the set \( S(1, a) = \{1, a\} \) is not an ideal of \( S \). Hence we know that \( S(u, v) \) may not be an ideal of \( S \) in general.

**Theorem 3.8.** Let \( S \) satisfy the left self-distributive law under \( \ast \), that is, \( x \ast (y \ast z) = (x \ast y) \ast (x \ast z) \) for all \( x, y, z \in S \). For any \( u, v \in S \), the set \( S(u, v) \) is an ideal of \( S \).

**Proof.** Let \( x \in S \) and \( a, b \in S(u, v) \). Then

\[
u \ast (v \ast (x \ast a)) = (u \ast (v \ast x)) \ast (u \ast (v \ast a)) = (u \ast (v \ast x)) \ast 1 = 1,
u \ast (v \ast ((a \ast (b \ast x)) \ast x)) = (u \ast (v \ast ((a \ast (b \ast x))))) \ast (u \ast (v \ast x))
\]

\[
= (u \ast (v \ast a)) \ast (u \ast (v \ast (b \ast x))) \ast (u \ast (v \ast x))
\]

\[
= (1 \ast (u \ast (v \ast b)) \ast (u \ast (v \ast x))) \ast (u \ast (v \ast x))
\]

\[
= (u \ast (v \ast x)) \ast (u \ast (v \ast x)) = 1.
\] (3.5)

Hence \( x \ast a \in S(u, v) \) and \( (a \ast (b \ast x)) \ast x \in S(u, v) \), which shows that \( S(u, v) \) is an ideal of \( S \).

**Lemma 3.9.** Let \( S \) be an implicative semigroup. If \( y \in S \) satisfies \( y \ast z = 1 \) for all \( z \in S \), then \( S(x, y) = S = S(y, x) \) for all \( x \in S \).

**Proof.** The proof is straightforward.

**Example 3.10.** Let \( S := \{1, a, b, c, d\} \) be an implicative semigroup with Cayley tables (Tables 3.3 and 3.4) and Hasse diagram (Figure 3.2) as follows:

**Table 3.3**

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It is easy to check that $S$ satisfies the left self-distributive law under $\ast$, that is, $x \ast (y \ast z) = (x \ast y) \ast (x \ast z)$ for all $x, y, z \in S$. By Lemma 3.9 we have $S(x, d) = S(d, x) = S$ for all $x \in S$. Furthermore we know that $S(1, 1) = \{1\}$, $S(1, a) = S(a, 1) = S(a, a) = S(a, b) = \{1, a\}$, $S(1, b) = S(b, 1) = S(b, b) = \{1, b\}$, $S(1, c) = S(a, c) = S(c, 1) = S(c, a) = S(c, c) = \{1, a, c\}$, $S(b, a) = \{1, a, b\}$, and $S(c, b) = S$ are ideals of $S$.

Using the set $S(u, v)$, we describe a characterization of ideals.

**Theorem 3.11.** Let $S$ be a commutative implicative semigroup and let $I$ be a non-empty subset of $S$. Then $I$ is an ideal of $S$ if and only if $S(u, v) \subseteq I$ for all $u, v \in I$.

**Proof.** Assume that $I$ is an ideal of $S$ and let $u, v \in I$. If $z \in S(u, v)$, then $u \ast (v \ast z) = 1 \in I$ and so $z = 1 \ast z = (u \ast (v \ast z)) \ast z \in I$ by (I2). Hence $S(u, v) \subseteq I$.

Conversely, suppose that $S(u, v) \subseteq I$ for all $u, v \in I$. Note that $1 \in S(u, v) \subseteq I$. Let $x, y, z \in S$ be such that $x \ast (y \ast z) \in I$ and $y \in I$. Since

$$(x \ast (y \ast z)) \ast (y \ast (x \ast z)) = (y \ast (x \ast z)) \ast (y \ast (x \ast z)) = 1,$$  

we have $x \ast z \in S(x \ast (y \ast z), y) \subseteq I$. Applying Theorem 3.7, we conclude that $I$ is an ideal of $S$. \hfill \Box

**Theorem 3.12.** Let $S$ be a commutative implicative semigroup. If $I$ is an ideal of $S$, then

$$I = \bigcup_{u, v \in I} S(u, v).$$  

**Proof.** Let $I$ be an ideal of $S$ and let $x \in I$. Obviously, $x \in S(x, 1)$ and so

$$I \subseteq \bigcup_{x \in I} S(x, 1) \subseteq \bigcup_{u, v \in I} S(u, v).$$  

Now let $y \in \bigcup_{u, v \in I} S(u, v)$. Then there exist $a, b \in I$ such that $y \in S(a, b)$. It follows from Theorem 3.11 that $y \in I$. Hence $\bigcup_{u, v \in I} S(u, v) \subseteq I$. This completes the proof. \hfill \Box

**Corollary 3.13.** If $I$ is an ideal of a commutative implicative semigroup $S$, then

$$I = \bigcup_{w \in I} S(w, 1).$$
References


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