STRONG UNIQUE CONTINUATION OF EIGENFUNCTIONS
FOR \( p \)-LAPLACIAN OPERATOR

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Abstract. We show the strong unique continuation property of the eigenfunctions for \( p \)-Laplacian operator in the case \( p < N \).

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1. Introduction. This paper is primarily concerned with the problem:

\[ -\text{div}(|\nabla u|^{p-2}\nabla u) + V|u|^{p-2}u = 0 \text{ in } \Omega, \tag{1.1} \]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) and the weight functions \( V \) is assumed to be not equivalent to zero and to lie in \( L^{N/p}(\Omega) \).

Also, we know that the unique continuation property is defined by a different form. In this work, we are interested to study a family of functions which enjoys the strong unique continuation property (SUCP), that is, functions besides possibly the zero functions has a zero of infinite order.

Definition 1.1. A function \( u \in L^p(\Omega) \) has a zero of infinite order in \( p \)-mean at \( x_0 \in \Omega \), if for each \( n \in \mathbb{N} \),

\[ \int_{|x-x_0| \leq R} |u|^p = 0(\mathbb{R}^n) \text{ as } R \to 0. \tag{1.2} \]

There is an extensive literature on unique continuation. We refer to the work of Jerison-Kenig on the unique continuation for Shrödinger operators (cf. [3]). The same work is done by Gossez and Figueiredo, but for linear elliptic operator in the case \( V \in L^{N/2} \), where \( N > 2 \), (cf. [1]). Also, Loulit extended this property to \( N = 2 \) by introducing Orlicz’s space, (cf. [2, 5]). In this work, we generalize this property for the \( p \)-Laplacian in the case \( V \in L^{N/p}(\Omega) \) and \( p < N \).

2. Strong unique continuation theorem. In this section, we proceed to establish the strong unique continuation property of the eigenfunctions for the \( p \)-Laplacian operator in the case \( V \in L^{N/p}(\Omega) \) and \( p < N \).

Theorem 2.1. Let \( u \in W^{1,p}_{\text{loc}}(\Omega) \) solution of (1.1). If \( u = 0 \) on a set \( E \) of positive measure, then \( u \) has a zero of infinite order in \( p \)-mean.
To prove this theorem we need the following lemmas.

**Lemma 2.2.** Let \( g \in W^{1,p}_0(\Omega) \) and \( V \in L^{N/p} \). Then for each \( \epsilon > 0 \) there exists a positive constant \( k_\epsilon \) such that

\[
\int_{\Omega} V |g|^p \leq \epsilon \int_{\Omega} |\nabla g|^p + k_\epsilon \int_{\Omega} |g|^p.
\]  

(2.1)

**Proof.** Set

\[
G = \left\{ x \in \Omega : \frac{\Omega}{V(x)} \geq c \right\}.
\]  

(2.2)

So

\[
\int_{\Omega} V |g|^p \leq \int_{G} V |g|^p + k \int_{\Omega} |g|^p.
\]  

(2.3)

By using the Hölder and Poincaré’s inequalities, we get

\[
\int_{\Omega} V |g|^p \leq c \left\| \chi_G V \right\|_{N/p} \int_{\Omega} |\nabla g|^p + k \int_{\Omega} |g|^p.
\]  

(2.4)

But \( \| \cdot \| \) is absolutely continuous. So, given \( \epsilon > 0 \), there exists \( k \) such that \( c \left\| \chi_G V \right\| \leq \epsilon \).

Which gives immediately the inequality (2.1).

**Lemma 2.3.** Let \( B_r \) and \( B_{2r} \) be two concentric balls contained in \( \Omega \). Then

\[
\int_{B_r} |\nabla u|^p \leq \frac{c}{r^p} \int_{B_{2r}} |u|^p,
\]  

(2.5)

where the constant \( c \) does not depend on \( r \).

**Proof.** Take \( \varphi \in C_0^\infty(\Omega) \), with \( \text{supp } \varphi \subset B_{2r} \), \( \varphi(x) = 1 \) for \( x \in B_r \) and \( |\nabla \varphi| \leq c/r \). Using \( \varphi^pu \) as a test function in (1.1), we get

\[
\int_{B_{2r}} -\text{div}(|\nabla u|^{p-2}\nabla u)\varphi^pu + \int_{B_{2r}} V|u|^{p-2}u\varphi^pu = 0.
\]  

(2.6)

So

\[
\int_{B_{2r}} |\nabla u|^p \varphi^p = -p \int_{B_{2r}} |\nabla u|^{p-2} \varphi^p \nabla u \cdot \nabla (\varphi u) - \int_{B_{2r}} V\varphi u|^p.
\]  

(2.7)

Using Young’s inequalities for \( ((p-1)/p) + 1/p = 1 \), we can estimate the first integral in the right-hand side of (2.7) by

\[
(p-1)e^{p/(p-1)} \int_{B_{2r}} |\nabla u|^p + \epsilon^{-p} \epsilon \int_{B_{2r}} |\nabla \varphi|^p |u|^p.
\]  

(2.8)

Also by the result of Lemma 2.2, we can estimate the second integral in the right-hand side of (2.7) by

\[
\epsilon \int_{B_{2r}} |\nabla (\varphi u)|^p + c_\epsilon \int_{B_{2r}} |\varphi u|^p.
\]  

(2.9)

Using these estimates in (2.7), we have

\[
\int_{B_{2r}} |\nabla u|^p \varphi^p \leq ((p-1)e^{p/(p-1)} + \epsilon) \int_{B_{2r}} |\nabla u|^p \varphi^p
\]

\[
+ (\epsilon^{-p} + \epsilon) \epsilon \int_{B_{2r}} |u|^p |\nabla \varphi|^p + c_\epsilon \int_{B_{2r}} |u|^p |\varphi|^p.
\]  

(2.10)

Using the fact that \( |\nabla \varphi| \leq c/r \), \( |\varphi| \leq c/r \), and \( \varphi = 1 \) in \( B_r \), we have immediately inequality (2.5).
**Lemma 2.4.** Let \( u \in W^{1,1}(B_r) \), where \( B_r \) is the ball of radius \( r \) in \( \mathbb{R}^N \) and let \( E = \{ x \in B_r : u(x) = 0 \} \). Then there exists a constant \( \beta \) depending only on \( N \) such that

\[
\int_A |u| \leq \beta \frac{r^N}{|E|} |A|^{1/N} \int_{B_r} |\nabla u|^{(2.11)}
\]

for all ball \( B_r, u \) as above and all measurable sets \( A \subset B_r \).

To prove this lemma see [4].

**Proof of Theorem 2.1.** We know that almost every point of \( E \) is a point of density of \( E \). Let \( x_0 \in E \) be such a point. This means that

\[
\lim_{r \to 0} \frac{|E \cap B_r|}{|B_r|} = 1,
\]

where \( B_r \) denotes the ball of radius \( r \) centered at \( x_0 \) and \( |S| \) denotes the Lebesgue measure of a set \( S \). So, given \( \epsilon > 0 \) there is an \( r_0 = r_0(\epsilon) \) such that

\[
\frac{|E^c \cap B_r|}{|B_r|} < \epsilon, \quad \frac{|E \cap B_r|}{|B_r|} > 1 - \epsilon \quad \text{for } r \leq r_0,
\]

(2.13)

where \( E^c \) denotes the complement of the set \( E \). Taking \( r_0 \) smaller, if necessary, we can assume \( B_{r_0} \subset \Omega \). Since \( u = 0 \) on \( E \), by Lemma 2.4 and (2.13) we have

\[
\int_{B_r} |u|^p = \int_{B_r \cap E^c} |u|^p \leq \beta \frac{r^N}{|E \cap B_r|} |E \cap B_r|^{1/N} \int_{B_r} |\nabla (u)^p| \leq p \beta \frac{r^N}{|B_r|} \frac{\epsilon^{1/N}}{1 - \epsilon} \int_{B_r} |u|^{p-1} |\nabla u|.
\]

(2.14)

By Hölder’s inequality

\[
\int_{B_r} |u|^p \leq c \frac{r^{1/N}}{1 - \epsilon} r \left( \int_{B_r} |\nabla u|^p \right)^{1/p} \left( \int_{B_r} |u|^p \right)^{(p-1)/p},
\]

(2.15)

and by using the Young’s inequality, we get

\[
\int_{B_r} |u|^p \leq c \frac{r^{1/N}}{1 - \epsilon} r^{p-1} \int_{B_r} |\nabla u|^p + \frac{p-1}{r} \int_{B_r} |u|^p.
\]

(2.16)

Finally, by Lemma 2.3, we have

\[
\int_{B_r} |u|^p \leq c \frac{r^{1/N}}{1 - \epsilon} \int_{B_{2r}} |u|^p,
\]

where \( c \) is independent of \( \epsilon \) and of \( r \) as \( r \to 0 \).
Now let us introduce the following functions:

\[ f(r) = \int_{B_r} |u|^p. \]  

(2.18)

And let us fix \( n \in \mathbb{N} \), choose \( \epsilon > 0 \) such that \((c\epsilon^{1/N})/(1 - \epsilon) \leq 2^{-n}\). Observe that consequently \( r_0 \) depends on \( n \). Then (2.17) can be written as

\[ f(r) \leq 2^{-n} f(2r) \quad \text{for} \ r \leq r_0. \]  

(2.19)

Iterating (2.19), we get

\[ f(\rho) \leq 2^{-kn} f(2^k \rho), \quad \text{if} \ 2^{k-1} \rho \leq r_0. \]  

(2.20)

Now given \( 0 < r < r_0(n) \) and choose \( k \in \mathbb{N} \) such that

\[ 2^{-k} r_0 \leq r \leq 2^{-k+1} r_0. \]  

(2.21)

From (2.20), we obtain

\[ f(r) \leq 2^{-kn} f(2^k r) \leq 2^{-kn} f(2r_0). \]  

(2.22)

Since \( 2^{-k} \leq r/r_0 \), we finally obtain

\[ f(r) \leq \left( \frac{r}{r_0} \right)^n f(2r_0), \]  

(2.23)

which shows that \( x_0 \) is a zero infinite order in \( p \)-mean.

\[ \square \]

REFERENCES


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