ON SOLUTIONS OF THE GOŁĄB-SCHINZEL EQUATION

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ABSTRACT. We determine the solutions $f : (0, \infty) \to [0, \infty)$ of the functional equation $f(x + f(x)y) = f(x)f(y)$ that are continuous at a point $a > 0$ such that $f(a) > 0$. This is a partial solution of a problem raised by Brzdek.

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The well-known Gołąb-Schinzel functional equation

$$f(x + f(x)y) = f(x)f(y)$$

has been studied by many authors (cf. [1, 3, 5, 7, 10]) in many classes of functions. Recently Aczél and Schwaiger [2], motivated by a problem of Kahlig, solved the following conditional version of (1)

$$f(x + f(x)y) = f(x)f(y)$$

for $x \geq 0, y \geq 0$, (2)

in the class of continuous functions $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Some further conditional generalizations of (1) have been considered by Reich [9] (see also [8] and Brzdek [4]).

At the 38th International Symposium on Functional Equations (Noszvaj, Hungary, June 11–17, 2000) Brzdek raised, among others, the problem (see [6]) of solving the equation

$$f(x + f(x)y) = f(x)f(y),$$

whenever $x, y, x + f(x)y \in \mathbb{R}_+$, (3)

in the class of functions $f : \mathbb{R}_+ \to \mathbb{R}$ that are continuous at a point, where $\mathbb{R}_+ = (0, \infty)$. We give a partial solution to the problem, namely we determine the solutions $f : \mathbb{R}_+ \to [0, \infty)$ of (3) that are continuous at a point $a \in \mathbb{R}_+$ such that $f(a) > 0$. Note that actually equations (1) and (3) have the same solutions in the class of functions $f : \mathbb{R}_+ \to [0, \infty]$.

From now on we assume that $f : \mathbb{R}_+ \to [0, \infty)$ is a solution of (3), continuous at a point $a \in \mathbb{R}_+$ such that $f(a) > 0$.

We start with some lemmas.

**Lemma 1.** Suppose that $y_2 > y_1 > 0$ and $f(y_1) = f(y_2) > 0$. Then

(a) $f(t + (y_2 - y_1)) = f(t)$ for $t \geq y_1$;

(b) for every $z > 0$ such that $f(z) > 0$,

$$f(t + f(z)(y_2 - y_1)) = f(t)$$

for $t \geq z + y_1f(z)$; (4)
(c) If \( z_1, z_2 > 0 \) and \( f(z_2) > f(z_1) > 0 \), then
\[
f(t + (f(z_2) - f(z_1))(y_2 - y_1)) = f(t) \quad \text{for} \ t \geq \max \{ z_1 + y_1 f(z_1), z_2 + y_1 f(z_2) \}.
\] (5)

**Proof.** (a) We argue in the same way as in [2, 7]. Namely, for \( t \geq y_1 \), by (3) we have
\[
f(t + (y_2 - y_1)) = f\left(y_2 + \frac{t - y_1}{f(y_1)} f(y_1)\right) = f\left(y_2 + \frac{t - y_1}{f(y_1)} f(y_2)\right)
\]
\[
= f(y_2) f\left(\frac{t - y_1}{f(y_1)}\right) = f(y_1) f\left(\frac{t - y_1}{f(y_1)}\right)
\]
\[
= f\left(y_1 + \frac{t - y_1}{f(y_1)} f(y_1)\right) = f(t).
\] (6)

(b) For every \( z > 0 \) such that \( f(z) > 0 \) we have
\[
f(z + y_1 f(z)) = f(z) f(y_1) = f(z) f(y_2) = f(z + y_2 f(z))
\] (7)
and consequently by (a) (with \( y_1 \) and \( y_2 \) replaced by \( z + y_1 f(z) \) and \( z + y_2 f(z) \))
\[
f(t) = f\left[t + (z + y_2 f(z) - z - y_1 f(z))\right] = f(t + f(z)(y_2 - y_1))
\] (8)
for \( t \geq z + y_1 f(z) \).

(c) Since \( (f(z_2) - f(z_1))(y_2 - y_1) > 0 \), \( t + (f(z_2) - f(z_1))(y_2 - y_1) \geq \max\{z_1 + y_1 f(z_1), z_2 + y_1 f(z_2)\} \) for \( t \geq \max\{z_1 + y_1 f(z_1), z_2 + y_1 f(z_2)\} \). Thus using (b) twice, for \( z = z_1 \) and \( z = z_2 \) (the first time with \( t \) replaced by \( t + (f(z_2) - f(z_1))(y_2 - y_1) \)), we have
\[
f(t + (f(z_2) - f(z_1))(y_2 - y_1))
\]
\[
= f\left[t + (f(z_2) - f(z_1))(y_2 - y_1) + f(z_1) (y_2 - y_1)\right]
\]
\[
= f(t + f(z_2)(y_2 - y_1)) = f(t)
\] (9)
for \( t \geq \max\{z_1 + y_1 f(z_1), z_2 + y_1 f(z_2)\} \). \( \square \)

**Lemma 2.** Let \( y_2 > y_1 > 0 \) and \( f(y_1) = f(y_2) > 0 \). Then there exists \( x_0 > 0 \) such that for every \( a > 0 \) there is \( c \in (0, a) \) with \( f(t + c) = f(t) \) for \( t \geq x_0 \).

**Proof.** First suppose that there is a neighbourhood \( U = (a - \delta, a + \delta) \) of \( a \) on which \( f \) is constant. Then for every \( x \in U \) such that \( a < x \), from Lemma 1(a), we get
\[
f(t + (x - a)) = f(t) \quad \text{for} \ t \geq a.
\] (10)
Thus it is enough to take \( x_0 = a \).

Now assume that there does not exist any neighbourhood of \( a \) on which \( f \) is constant. Take \( \varepsilon \in (0, f(a)) \). The continuity of \( f \) at \( a \) implies that there exists \( \delta \in (0, 1) \) such that for every \( x \in U_1 = (a - \delta, a + \delta) \) we have \( f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon) \). Take \( x_1, x_2 \in U_1 \) such that \( f(x_1) < f(x_2) \). Then \( f(x_2) - f(x_1) < 2\varepsilon \). From \( \varepsilon < f(a) \) we infer \( f(x_1) > 0 \) and by Lemma 1(c) we get
\[
f(t + (f(x_2) - f(x_1))(y_2 - y_1)) = f(t) \quad \text{for} \ t \geq \max\{x_1 + y_1 f(x_1), x_2 + y_1 f(x_2)\}.
\] (11)
Next by a suitable choice of \( \varepsilon \) the value \( c := (f(x_2) - f(x_1))(y_2 - y_1) \) can be made arbitrarily small. Moreover, \( x_1, x_2 > a + 1 \) and \( f(x_1), f(x_2) < f(a) + \varepsilon < 2f(a) \), which means that \( f(t + c) = f(t) \) for \( t \geq x_0 := a + 1 + y_1 f(a) \). This completes the proof. \( \square \)
LEMMA 3. If for some $y_2 > y_1 > 0$, $f(y_1) = f(y_2) > 0$, then for every $\varepsilon > 0$ and $\epsilon > 0$ there is $c \in (0, \epsilon)$ with $f(t + c) = f(t)$ for $t \geq \varepsilon$.

PROOF. By Lemma 2 there exists $x_0 > 0$ such that for arbitrarily small $c > 0$

$$f(t + c) = f(t) \quad \text{for} \quad t \geq x_0. \quad (12)$$

By induction, from Lemma 1(a), we get $f(y_1) = f(y_1 + n(y_2 - y_1))$ for any positive integer $n$. Consequently there exists $x_1 \in [x_0, \infty)$ with $f(x_1) = f(y_1)$.

Put $B = \{x > x_0 : f(x) > 0\}$. Clearly $x_1 \in B$. Thus (12) implies that $B \cap A \neq \emptyset$ for every nontrivial interval $A \subset [x_0, \infty)$. Define a function $f_1 : [0, \infty) \to [x_0, \infty)$ by

$$f_1(x) = x_1 + x f(x_1). \quad (13)$$

Note that $f_1$ is increasing. Let $\varepsilon > 0$ and $y_0 \in B \cap (f_1(0), f_1(\varepsilon)) \neq \emptyset$. By the continuity of $f_1$ there exists $z_0 \in (0, \varepsilon)$ such that $f_1(z_0) = y_0$. Take $d > 0$ with $f(t + d) = f(t)$ for $t \geq x_0$. Then

$$f(y_0) = f(y_0 + d) \neq 0. \quad (14)$$

The form of the function $f_1$ implies that there exists $z_1 > z_0$ such that $f_1(z_1) = y_0 + d$. Note that (14) yields

$$f(x_1 + z_0 f(x_1)) = f(f_1(z_0))$$

$$= f(y_0) = f(y_0 + d) = f(f_1(z_1))$$

$$= f(x_1 + z_1 f(x_1)) \neq 0. \quad (15)$$

Further by (3)

$$f(x_1) f(z_0) = f(x_1) f(z_1) \neq 0, \quad (16)$$

and consequently $f(z_0) = f(z_1) > 0$. Hence, in view of Lemma 1(a), we infer that

$$f(t + (z_1 - z_0)) = f(t) \quad \text{for} \quad t \geq z_0. \quad (17)$$

This completes the proof, because $\varepsilon > z_0$ and, choosing sufficiently small $d$, we can make $c := (z_1 - z_0)$ arbitrarily small. \hfill \Box

LEMMA 4. If there exist $y_2 > y_1 > 0$ such that $f(y_1) = f(y_2) > 0$, then $f \equiv 1$.

PROOF. First we show that $f(x) = f(a) = b$ for $x \in \mathbb{R}_+$. For the proof by contradiction suppose that there exists $t_0 > 0$ with $f(t_0) \neq f(a)$. Put

$$\varepsilon_0 = |f(t_0) - f(a)|. \quad (18)$$

The continuity of $f$ at $a$ implies that there exists $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon_0$. By Lemma 3 there exists $y_0 > 0$ such that $|y_0 - a| < \delta$ and $f(y_0) = f(t_0)$, which means that $|f(t_0) - f(a)| < \varepsilon_0$, contrary to (18). Thus we have proved that $f \equiv b$. Clearly from (3) we get $b = f(a) = f(a + f(a)) = f(a)^2 = b^2$ and consequently $b = 1$. This completes the proof. \hfill \Box

LEMMA 5. If $f$ is nonconstant then $(f(x) - 1)/x$ is constant for all $x > 0$ with $f(x) > 0$. 


Moreover, by the continuity of $f$, $f(x) > 0$ for every $x > 0$. Clearly, $\delta = \inf_{B} f(b) < 0$. Thus, by Lemma 4, $f = 1$, a contradiction.

**Remark 6.** If we denote the constant in Lemma 5 by $c$, then from Lemma 5 we get $f(x) \in \{cx+1, 0\}$ for every $x > 0$. In the case $c < 0$ we have $f(x) = 0$ for every $x \geq -1/c$ (because $f \geq 0$).

**Lemma 7.** Suppose that $f$ is nonconstant. Then,

(a) in the case $c := (f(a) - 1)/a < 0$, $f(x) = cx + 1$ for $x \in (0, -1/c)$;

(b) in the case $c := (f(a) - 1)/a > 0$, $f(x) = cx + 1$ for $x > 0$.

**Proof.** The continuity of $f$ at $a$ implies that there exists $\delta \in (0, a)$ such that $f(x) > 0$ for every $x \in U = (a - \delta, a + \delta)$. Thus, by Remark 6, $f(x) = cx + 1$ for $x \in U$.

Let $I = (a, -1/c)$ if $c < 0$ and $I = (a, \infty)$ if $c > 0$. Put $B_{1} := \{x \in (0, a) : f(x) = 0\}$, $B_{2} := \{x \in I : f(x) = 0\}$, $B = B_{1} \cup B_{2},$

\[
d_{1} := \begin{cases}
sup B_{1} & \text{if } B_{1} \neq \emptyset, \\
a - \delta & \text{if } B_{1} = \emptyset,
\end{cases}
\quad d_{2} := \begin{cases}
\inf B_{2} & \text{if } B_{2} \neq \emptyset, \\
a + \delta & \text{if } B_{2} = \emptyset.
\end{cases}
\]

Clearly $f(x) > 0$ on the interval $A = (d_{1}, d_{2}) \supset (a - \delta, a + \delta)$.

(a) For the proof by contradiction suppose that there exists $b_{1} \in (0, -1/c)$ with $f(b_{1}) = 0$. Notice that $d_{2} < -1/c$. Indeed, if $B_{2} \neq \emptyset$ then, since $B_{2} \subset (a, -1/c)$, so $\inf B_{2} < -1/c$. If not, then from Remark 6 we have that $a + \delta < -1/c$. Consequently $d_{2} < -1/c$. Thus $cd_{2} > -1$ and consequently $\delta + \delta cd_{2} > 0$. Take $b \in B$ and $z \in A$ such that $|z - b| < \delta + \delta cd_{2}$. Define functions $h, g : U \to \mathbb{R}$ by

\[
h(x) = x + zf(x) \quad \text{for} \ x \in U, \\
g(x) = x + bf(x) \quad \text{for} \ x \in U.
\]

By the continuity of $f$ on $U$, $h$ is continuous. Next, since $z < d_{2}$, so $cz > cd_{2}$ and $\delta + \delta cz > \delta + \delta cd_{2} > 0$. Hence

\[
h(a) - h(a - \delta) = a + z(c(a + 1) + 1) - a - \delta - z[\delta + \delta cz] > 0,
\]

\[
h(a + \delta) - h(a) = a + \delta + z[\delta + \delta cz] - a - z(c + 1) > 0.
\]

Moreover $1 > c(a + 1) = f(a) > 0$, whence

\[
|h(a) - g(a)| = |a + z(c(a + 1) + 1) - a - b(c(a + 1))| \\
= |z - b||c(a + 1)| < |z - b| < \delta + \delta cd_{2} < \delta + \delta cz.
\]

From (23) and (24) we obtain

\[
h(a - \delta) < g(a) < h(a + \delta).
\]
The continuity of $h$ implies that there exists $x_0 \in (a - \delta, a + \delta)$ such that $h(x_0) = g(a)$. Since $a, x_0, z \in A$ and $b \in B$, so we have

$$0 \neq f(x_0)f(z) = f(x_0 + zf(x_0)) = f(h(x_0)) = f(g(a)) = f(a + bf(a)) = f(a)f(b) = 0.$$  \hspace{1cm} (26)

This contradiction ends the proof of (a).

(b) For the proof by contradiction suppose that $f(b_1) = 0$ for some $b_1 > 0$. Since $ca + 1 = f(a) > 0$, there are $b \in B$ and $z \in A$ such that $|z - b| < \delta/(ca + 1)$. Define functions $h, g : U \to \mathbb{R}$ in the same way as in the proof of (a). Then (23) holds and

$$|h(a) - g(a)| = |z - b|(ca + 1) < \frac{\delta}{ca + 1} (ca + 1) = \delta < \delta + \delta cz.$$  \hspace{1cm} (27)

Hence

$$h(a - \delta) < g(a) < h(a + \delta).$$  \hspace{1cm} (28)

We obtain a contradiction in a similar way as in the proof of (a).

**Lemma 8.** If $c := (f(a) - 1)/a = 0$, then $f(x) = 1$ for $x > 0$.

**Proof.** The continuity of $f$ at $a$ implies that there exists $\delta > 0$ such that $f(x) > 0$ for every $x \in [a - \delta, a + \delta]$. Thus, by Lemma 5 and Remark 6, $f(x) = cx + 1 = 1$ for every $x \in [a - \delta, a + \delta]$. Hence Lemma 4 implies that $f(x) = 1$ for every $x > 0$.

Finally from Lemmas 7 and 8 and Remark 6 we get the following theorem.

**Theorem 9.** If a function $f : \mathbb{R}^+ \to [0, \infty)$ is continuous at a point $a$ such that $f(a) \neq 0$ and satisfies (3), then

$$f(x) = \max\{cx + 1, 0\} \quad \forall x \in \mathbb{R}^+.$$  \hspace{1cm} (29)

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**References**


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