DEGREE OF APPROXIMATION OF CONJUGATE OF A FUNCTION BELONGING TO Lip(ξ(t),p) CLASS BY MATRIX SUMMABILITY MEANS OF CONJUGATE FOURIER SERIES

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Abstract. We determine the degree of approximation of conjugate of a function belonging to Lip(ξ(t),p) class by matrix summability means of a conjugate series of a Fourier series.

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1. Introduction. Bernstein [2], Alexits [1], Sahney and Goel [14], and Chandra [4] have determined the degree of approximation of a function belonging to Lipα by (C,1), (C,δ), (N,pn), and (N,pn) means of its Fourier series. Working in the same direction Sahney and Rao [15] and Khan [6] have studied the degree of approximation of functions belonging to Lip(α,p) by (N,pn) and (N,p,q) means, respectively. The (N,p,q) summability reduces to (N,pn) summability for qn = 1 for all n, and to (N,qn) means when pn = 1 for all n. After quite a good amount of work on degree of approximation of function by different summability means of its Fourier series, for the first time in 1981, Qureshi [12, 13] discussed the degree of approximation of conjugate of a function belonging to Lipα and Lip(α,p) by (N,pn) means of conjugate Fourier series. But nothing seems to have been done so far to obtain the degree of approximation of conjugate of a function belonging to Lipξ(t),p class by matrix means of conjugate Fourier series. The Lip(ξ(t),p) class is a generalization of Lipα and Lip(α,p). Matrix means includes as special cases the method of (C,1), (C,δ), (N,pn), (N,pn), and (N,p,q) means. In an attempt to make an advance study in this direction, we, in this paper, establish a theorem on degree of approximation of conjugate of a function of Lip(ξ(t),p) class by matrix summability means of conjugate series of a Fourier series then both the results of Qureshi [12, 13] come out as particular cases of our theorem.

2. Definitions and notations. Let f be periodic with period 2π and integrable in the Lebesgue sense. Let its Fourier series be given by

\[ f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \]  

(2.1)

The conjugate series of (2.1) is given by

\[ \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) = \sum_{n=1}^{\infty} B_n(x). \]  

(2.2)
Let \( \{ p_n \} \) be a nonnegative nonincreasing generating sequence for \((N, p_n)\) method such that
\[
P_n = P(n) = p_0 + p_1 + p_2 + \cdots + p_n \to \infty, \quad \text{as } n \to \infty.
\] (2.3)

Let \( T = (a_{n,k}) \) be an infinite triangular matrix satisfying the Silverman Toeplitz [16], that is,
\[
\sum_{k=0}^{n} a_{n,k} \to 1, \quad \text{as } n \to \infty, \quad a_{n,k} = 0, \quad \text{for } k > n,
\] (2.4)

\[
\sum_{k=0}^{n} |a_{n,k}| \leq M, \quad \text{a finite constant.}
\]

Let \( \sum_{m=0}^{\infty} u_m \) be an infinite series such that
\[
s_k = u_0 + u_1 + u_2 + \cdots + u_k = \sum_{m=0}^{k} u_m,
\] (2.5)

that is, \( s_k \) denotes the \( k \)th partial sum of the series \( \sum_{m=0}^{\infty} u_m \).

The sequence-to-sequence transformation
\[
t_n = \sum_{k=0}^{n} a_{n,k} s_k = \sum_{k=0}^{n} a_{n,n-k}s_{n-k}
\] (2.6)

defines the sequence \( \{ t_n \} \) of matrix means of the sequence \( \{ s_n \} \) generated by the sequence of the coefficients \( (a_{n,k}) \). The series \( \sum u_n \) is said to be summable to the sum “\( S \)” by matrix method if \( \lim_{n \to \infty} t_n \) exists and equal to \( S \) (see Zygmund [17]) and we write
\[
t_n \to S(T), \quad \text{as } n \to \infty.
\] (2.7)

2.1. Particular cases. Seven important cases of matrix means are
(1) \((C,1)\) means when \( a_{n,k} = 1/(n+1) \).
(2) Harmonic means when \( a_{n,k} = 1/(n-k+1) \log n \).
(3) \((C,\delta)\) means when \( a_{n,k} = (n-k+\delta-1)/(n\delta) \).
(4) \((H,p)\) means when \( a_{n,k} = 1/\log^{p-1}(n+1) \prod_{q=0}^{p-1} \log^{q}(k+1) \).
(5) Nörlund means [11] when \( a_{n,k} = p_{n-k}/P_n \) where \( P_n = \sum_{k=0}^{n} p_k, q_n = 1 \) for all \( n \).
(6) Riesz means \((\bar{N}, p_n)\) [5] when \( a_{n,k} = p_k/P_n, q_n = 1 \) for all \( n \).
(7) Generalized Nörlund mean \((N,p,q)\) [3] when \( a_{n,k} = p_{n-k}q_k/R_n \) where
\[
R_n = \sum_{k=0}^{n} p_k q_{n-k}.
\] (2.8)

In particular cases (5), (6), and (7), \( \{ p_n \} \) and \( \{ q_n \} \) are two nonnegative monotonic nonincreasing sequences of real constants.

We define the norm
\[
\| f \|_p = \left\{ \int_{0}^{2\pi} |f(x)|^p \, dx \right\}^{1/p}, \quad p \geq 1
\] (2.9)
and let the degree of approximation be given by (see Zygmund [17])

\[ E_n(f) = \min_{\mathcal{T}_n} ||f - \mathcal{T}_n||_p, \]  

(2.10)

where \( \mathcal{T}_n(x) \) is some \( n \)th degree trigonometric polynomial.

A function \( f \in \text{Lip} \alpha \) if

\[ f(x + t) - f(x) = O(t^\alpha), \quad \text{for } 0 < \alpha \leq 1 \]  

(2.11)

and \( f \in \text{Lip}(\alpha, p) \) if

\[ \left\{ \int_0^{2\pi} |f(x + t) - f(x)|^p \, dx \right\}^{1/p} = O(t^\alpha), \quad 0 < \alpha \leq 1, \quad p \geq 1 \]  

(2.12)

(see [10, Definition 5.38]).

Given a positive increasing function \( \xi(t) \) and an integer \( p > 1 \), then \( f(x) \in \text{Lip}(\xi(t), p) \) if

\[ \left\{ \int_0^{2\pi} |f(x + t) - f(x)|^p \, dx \right\}^{1/p} = O(\xi(t)), \quad \text{(see [8])} \]  

(2.13)

In case \( \xi(t) = t^\alpha \), we notice that \( \text{Lip}(\xi(t), p) \) class coincides with known \( \text{Lip}(\alpha, p) \) class [10].

We use the following notations:

\[ \psi(t) = f(x + t) - f(x - t), \]

\[ A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-k}, \]

\[ \tau = \text{integral part of } \frac{1}{t} = \left\lfloor \frac{1}{t} \right\rfloor, \]  

(2.14)

\[ K_n(t) = \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\cos(n-k-1/2)t}{\sin t/2}. \]

3. Known theorems. Qureshi [12] has proved the following theorem.

**Theorem 3.1.** If the sequence \( \{p_n\} \) satisfies the following conditions:

\[ n \left| p_n \right| < C \left| P_n \right|, \quad \sum_{k=1}^{n} k \left| p_k - p_{k-1} \right| < C \left| P_n \right|, \]  

(3.1)

then the degree of approximation of a function \( \hat{f}(x) \), conjugate to a periodic function \( f \) with period \( 2\pi \) and belonging to the class \( \text{Lip} \alpha, 0 < \alpha < 1 \) by Nörlund means of its conjugate series, is given by

\[ |\hat{f}(x) - \tilde{\mathcal{I}}_n(x)| = O\left(\frac{1}{P_n} \sum_{k=1}^{n} \frac{P_k}{k^{\alpha+1}} \right), \]  

(3.2)

where \( \tilde{\mathcal{I}}_n(x) \) are the \( (N, p_n) \) means of series (2.2).
Qureshi [13] has proved another theorem in the following form.

**Theorem 3.2.** If \( f(x) \) is periodic and belongs to the class \( \text{Lip}(\alpha,p) \) for \( 0 < \alpha \leq 1 \), and if the sequence \( \{p_n\} \) is as defined in (2.3) with other requirements therein and if

\[
\int_1^n \left( \frac{(p(y)^q)}{y^{q\alpha+2-\delta q-\delta}} \right) = O \left( \frac{p(n)}{n^{\alpha-1/q-\delta-1}} \right),
\]

then

\[
\|\tilde{i}_n - \tilde{f}\|_p = O \left( \frac{1}{n^{\alpha-1/p}} \right),
\]

where \( \tilde{i}_n \) are the \((N,p_n)\) means of the series (2.2) and \( 1/p + 1/q = 1 \) such that \( 1 \leq p \leq \infty \).

4. Main theorem. Our object of this paper is to prove the following theorem.

**Theorem 4.1.** If \( T = (a_{n,k}) \) is an infinite regular triangular matrix such that the elements \( a_{n,k} \) is nonnegative and nondecreasing with \( k \), then the degree of approximation of a function \( \tilde{f}(x) \), conjugate to a \( 2\pi \)-periodic function \( f \) belonging to \( \text{Lip}(\xi(t),p) \) class by matrix summability means of its conjugate series is given by

\[
\|\tilde{i}_n(x) - \tilde{f}(x)\| = O \left( \xi \left( \frac{1}{n} \right) n^{1/p} \right)
\]

provided \( \xi(t) \) satisfies the following conditions:

\[
\left\{ \int_0^{1/n} \left( \frac{t}{\xi(t)} \right)^p \psi(t) \right\}^{1/p} = O \left( \frac{1}{n} \right),
\]

\[
\left\{ \int_0^{\pi} \left( \frac{t^{\delta}}{\xi(t)} \right)^p \psi(t) \right\}^{1/p} = O(n^{\delta}),
\]

where \( \delta \) is an arbitrary number such that \( q(1 - \delta) - 1 > 0 \), conditions (4.2) and (4.3) hold uniformly in \( x \),

\[
\tilde{i}_n(x) = \sum_{k=0}^{n} a_{n,n-k} \tilde{f}_{n-k}(x),
\]

that is, matrix means of conjugate Fourier series (2.2), \( 1/p + 1/q = 1 \), such that \( 1 \leq p \leq \infty \) and

\[
\tilde{f}(x) = -\frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot \frac{1}{2} t dt.
\]

5. Lemmas. For the proof of our theorem the following lemmas are required.

**Lemma 5.1** [9]. If \( a_{n,k} \) is nonnegative and nonincreasing with \( k \), then for \( 0 \leq a \leq b \leq \infty \), \( a \leq t \leq \pi \) and any \( n \),

\[
\left| \sum_{k=a}^{b} a_{n,n-k} e^{i(n-k)t} \right| = O(A_{n,T}).
\]
**Lemma 5.2.** Under the conditions of Theorem 4.1 on \((a_{n,k})\) for \(0 < 1/n \leq t \leq \pi\),

\[
K_n(t) = O\left(\frac{An,\tau}{t}\right).
\]

**Proof.** Since for \(0 < 1/n \leq t \leq \pi\), \(\sin(t/2) < t\), therefore for \(t > 0\) and \(\tau \leq n\), we have,

\[
|K(t)| = \left| \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\cos(n-k-1/2)t}{\sin(t/2)} \right| \\
\leq \left| \frac{1}{2\pi} \Re \sum_{k=0}^{n} a_{n,n-k}e^{i(n-k-1/2)t} \frac{1}{\sin(t/2)} \right| \\
= O\left(\frac{1}{t} \left| \sum_{k=0}^{n} a_{n,n-k}e^{i(n-k)t} \right| |e^{-it/2}| \right) \\
= O\left(\frac{1}{t} \left| \sum_{k=0}^{n} a_{n,n-k}e^{i(n-k)t} \right| \right) \\
= O\left(\frac{An,\tau}{t}\right)
\]

by Lemma 5.1.

\[\square\]

6. **Proof of the main theorem.** Let \(s_n(x)\) denote the nth partial sum of series (2.2), then, following [7], we have

\[
s_n(x) - \left( -\frac{1}{2\pi} \int_{0}^{\pi} \psi(t) \cot \frac{1}{2} t dt \right) = \frac{1}{2\pi} \int_{0}^{\pi} \psi(t) \frac{\cos(n+1/2)t}{\sin(t/2)} dt.
\]

(6.1)

Now

\[
\sum_{k=0}^{n} a_{n,n-k} \left\{ s_{n-k}(x) - \left( -\frac{1}{2\pi} \int_{0}^{\pi} \psi(t) \cot \frac{1}{2} t dt \right) \right\} \\
= \frac{1}{2\pi} \int_{0}^{\pi} \psi(t) \sum_{k=0}^{n} a_{n,n-k} \frac{\cos(n-k-1/2)t}{\sin(t/2)} dt
\]

(6.2)

or

\[
\tilde{s}_n(x) - \tilde{f}(x) = \frac{1}{2\pi} \int_{0}^{\pi} \psi(t) \sum_{k=0}^{n} a_{n,n-k} \frac{\cos(n-k-1/2)t}{\sin(t/2)} dt \\
= \int_{0}^{\pi} \psi(t) \tilde{K}_n(t) dt \\
= \int_{0}^{1/n} \psi(t) \tilde{K}_n(t) dt + \int_{1/n}^{\pi} \psi(t) \tilde{K}_n(t) dt
\]

(6.3)

\[= I_1 + I_2.\]
Applying Hölder’s inequality and the fact that \( \psi(t) = W(\text{Lip} \xi(t), p) \), we get

\[
I_1 = \int_0^{1/n} \psi(t) \mathcal{K}_n(t) \, dt
\]

\[
\leq O \left[ \int_0^{1/n} \left( \frac{t |\psi(t)|}{\xi(t)} \right)^p \, dt \right]^{1/p} \left[ \int_0^{1/n} \left( \frac{\mathcal{K}_n(t) \xi(t)}{t} \right)^q \, dt \right]^{1/q}
\]

\[
= O \left( \frac{1}{n} \right) \left[ \int_0^{1/n} \left( \frac{\xi(t)}{t} \sum_{k=0}^n a_{n,n-k} \frac{\cos(n-k-1/2)t}{\sin(t/2)} \right)^q \, dt \right]^{1/q}
\]

\[
= O \left( \frac{1}{n} \right) \left[ \int_0^{1/n} \left( \frac{\xi(t)}{t^2} \right)^q \, dt \right]^{1/q}
\]

\[
= O \left( \frac{1}{n} \right) O \left( \frac{\xi \left( \frac{1}{n} \right)}{n} \right) \left[ \int_1^{1/n} \frac{dt}{t^{2q}} \right]^{1/q} \quad \text{by mean value theorem}
\]

\[
= O \left( \frac{1}{n} \right) O \left( \frac{\xi \left( \frac{1}{n} \right)}{n} \right) \left[ \int_1^{1/n} \left( \frac{t^{-2q+1}}{-2q+1} \right)^{1/n} \, dt \right]^{1/q}
\]

\[
= O \left( \frac{\xi \left( \frac{1}{n} \right)}{n} \right) O \left( n^{2-1/q} \right)
\]

\[
= O \left( \frac{\xi \left( \frac{1}{n} \right)}{n} \right) n^{1-1/q}
\]

\[
I_1 = O \left( \frac{\xi \left( \frac{1}{n} \right)}{n} \right) n^{1/p} \quad \text{(since } \frac{1}{p} + \frac{1}{q} = 1 \).
\]

Consider \( I_2 \)

\[
I_2 = \left[ \int_{1/n}^{\pi} \left( \frac{t^{-\delta} \psi(t)}{\xi(t)} \right)^p \, dt \right]^{1/p} \left[ \int_{1/n}^{\pi} \left( \frac{\mathcal{K}_n(t) \xi(t)}{t^{-\delta}} \right)^q \, dt \right]^{1/q}
\]

\[
= O \left[ \int_{1/n}^{\pi} \left( \frac{t^{-\delta} |\psi(t)|}{\xi(t)} \right)^p \, dt \right]^{1/p} O \left[ \int_{1/n}^{\pi} \left( \frac{\xi(t) A_{n,t}}{t^{-\delta+1}} \right)^q \, dt \right]^{1/q} \quad \text{by Lemma 5.2}
\]

\[
= O(n^\delta) \cdot O \left[ \int_{1/n}^{\pi} \left( \frac{\xi(t) A_{n,t}}{t^{-\delta+1}} \right)^q \, dt \right]^{1/q} \quad \text{by condition (4.3)}
\]

\[
= O(n^\delta) \cdot O \left[ \int_{1/n}^{\pi} \left( \frac{\xi \left( \frac{1}{n} \right) A_{n,n}}{y^{\delta-1}} \right)^q \, dy \right]^{1/q}
\]

\[
= O(n^\delta) \cdot O \left( \frac{\xi \left( \frac{1}{n} \right) A_{n,n}}{n} \right) \left[ \int_{1/n}^{\pi} \left( \frac{dy}{y^{q(\delta-1)+2}} \right) \right]^{1/q} \quad \text{by mean value theorem}
\]
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\[ I_2 = O \left( \frac{1}{n^1/p} \right) \quad \text{(since } \frac{1}{p} + \frac{1}{q} = 1) \]

(6.5)

By combining (6.3), (6.4), and (6.5) we have

\[ |\tilde{t}_n(x) - \tilde{f}(x)| = O \left( \frac{1}{n^1/p} \right), \quad (6.6) \]

therefore

\[ ||\tilde{t}_n(x) - \tilde{f}(x)||_p = O \left[ \left\{ \int_0^{2\pi} \left( \frac{1}{n^1/p} \right) dx \right\}^{1/p} \right] \]

\[ = O \left[ \left( \frac{1}{n^1/p} \right) \left( \int_0^{2\pi} dx \right)^{1/p} \right] \]

(6.7)

This completes the proof of the theorem.

7. Applications. The following corollaries can be derived from the main theorem.

**Corollary 7.1.** If \( \xi(t) = t^\alpha, \ 0 < \alpha \leq 1 \), then the degree of approximation of a function \( \tilde{f}(x) \), conjugate to \( 2\pi \)-periodic function \( f \) belonging to the class \( \text{Lip}(\alpha, p) \) is given by

\[ |\tilde{t}_n(x) - \tilde{f}(x)| = O \left( \frac{1}{n^{\alpha-1/p}} \right). \quad (7.1) \]

**Proof.** We have

\[ ||\tilde{t}_n(x) - \tilde{f}(x)||_p = O \left\{ \int_0^{2\pi} |\tilde{t}_n(x) - \tilde{f}(x)|^p dx \right\}^{1/p} \]

(7.2)

or

\[ \left( \frac{1}{n^1/p} \right)^p = O \left\{ \int_0^{2\pi} |\tilde{t}_n(x) - \tilde{f}(x)|^p dx \right\}^{1/p} \]

(7.3)

or

\[ O(1) = O \left\{ \int_0^{2\pi} |\tilde{t}_n(x) - \tilde{f}(x)|^p dx \right\}^{1/p} \quad \text{O} \left( \frac{1}{\xi(1/n)n^{1/p}} \right). \]

(7.4)

Hence

\[ |\tilde{t}_n(x) - \tilde{f}(x)| = O \left[ \xi \left( \frac{1}{n} \right) n^{1/p} \right] \quad (7.5) \]
for if not the right-hand side will be $O(1)$, therefore
\[
|\tilde{t}_n(x) - \tilde{f}(x)| = O\left(\frac{1}{n^{\alpha \frac{1}{p}}}\right) = O\left(\frac{1}{n^{\alpha - \frac{1}{p}}}\right).
\] (7.6)

This completes the proof.

**Corollary 7.2.** If $p \to \infty$ in Corollary 7.1, then for $0 < \alpha < 1$,
\[
|\tilde{t}_n(x) - \tilde{f}(x)|_p = O\left(\frac{1}{n^{\alpha}}\right).
\] (7.7)

**Remark 7.3.** An independent proof of Corollary 7.1 can be derived along the same lines as the theorem.

**8. Particular cases.** (1) If $a_{n,k} = p_{n-k}/p_n$, $\xi(t) = t^\alpha$, $0 < \alpha < 1$, $p \to \infty$ and using
\[
1/n^\alpha \leq 1/p_n \sum_{k=1}^n p_k/k^{\alpha + 1}
\]
(see [14, Lemma 1]), then the result of Qureshi [12] becomes the particular case of the main theorem.

(2) The result of Qureshi [13] becomes the particular case of our theorem if $(a_{n,k})$ is defined as in case (1) and $\xi(t) = t^\alpha$, $0 < \alpha \leq 1$.

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