A NEW COMBINATORIAL IDENTITY

JOSEPH SINYOR, TED SPEEVAK, and AKALU TEFERA

(Received 5 June 2000)

ABSTRACT. We prove a combinatorial identity which arose from considering the relation $r_p(x, y, z) = (x + y - z)^p - (x^p + y^p - z^p)$ in connection with Fermat's last theorem.

2000 Mathematics Subject Classification. Primary 05-02.

The following combinatorial identity:

$$\sum_{l' \leq l} \sum_{j' \leq j} \frac{1}{(m-l')} \binom{m+l'-j'}{2l'-j'+1} \binom{m-l'+j'-1}{j'} \binom{m-l'}{2(l-l')-(j-j')} \binom{m-l'}{j-j'} \quad (1)$$

$$= \frac{1}{2(m-l)} \binom{2m}{2l+1} \binom{2l+1}{j} = \frac{1}{(2l+1)} \binom{2m}{2l} \binom{2l+1}{j}$$

for all $m > l \geq 0$, where m , l , and j are nonnegative integers and $0 \leq j \leq 2l+1$, arose from considering

$$r_p(x, y, z) = (x + y - z)^p - (x^p + y^p - z^p) \quad (2)$$

in connection with Fermat's last theorem (FLT), which was proved in 1994 by Wiles and Taylor. Recall that FLT states that $x^p + y^p - z^p \neq 0$, where x , y , z , p are any nonzero integers and $p > 2$. We take, without loss of generality, that x , y , and z are relatively prime and p is prime. In general, $r_p(x, y, z)$ can be factored as $p(z-x)(z-y)(x+y)f_p(x, y, z)$ which are powers of p if $x^p + y^p - z^p = 0$. These factors result in the elementary Abel-Barlow relations known since the 1820's (see [2]).

However, the last factor $f_p(x, y, z)$ is

$$\sum_{l=0}^{m-1} \sum_{i=0}^{2l} \sum_{j=0}^i \frac{(-1)^{i-j}}{(m-l)} \binom{m+l-j}{2l-j+1} \binom{m-l+j-1}{j} x^{2l-i} y^i (z-x)^{m-l-1} (z-y)^{m-l-1} \quad (3)$$

$$= \begin{cases} p^{kp-1} d^p, & p \nmid xyz, \\ d^p, & p \mid xyz, \end{cases}$$

where $p = 2m+1 \geq 5$ and $k > 0$. This formulation of $f_p(x, y, z)$, which is believed to be novel, establishes the new identity. However, it appears to offer no new insights into a possible elementary proof of FLT.

To discover the identity, note that

$$r_p(x, y, z) = p \sum_{l=0}^{2m} \sum_{j=0}^{2m} \frac{(-1)^l}{p} \binom{p}{l} \binom{p-l}{j} x^j y^{p-j-l} z^l, \quad (4)$$

where $j+l \neq 0$.

Alternatively, we have

$$r_p(x, y, z) = p \sum_{l'=0}^m (z-x)^{m-l'} (z-y)^{m-l'} \sum_{j'=0}^{2l'+1} a_{j', m-l'} x^{2l'-j'+1} y^{j'}. \quad (5)$$

Equating (4) and (5) for a given j and l , we get the recurrence

$$a_{j, m-l} = \frac{1}{2(m-l)} \binom{2m}{2l+1} \binom{2l+1}{j} - \sum_{l' < l} \sum_{j' \leq j} a_{j', m-l'} \binom{m-l'}{2(l-l')-(j-j')} \binom{m-l'}{j-j'}. \quad (6)$$

Now,

$$a_{j, m-l} = \frac{1}{(m-l)} \binom{m+l-j}{2l-j+1} \binom{m-l+j-1}{j} \quad (7)$$

satisfies the recurrence (6). Substituting the expression for $a_{j, m-l}$ and rearranging, we obtain the new identity.

The authors have reviewed the literature, notably Gould [1] and Riordan [3] as well as the relevant journals since 1980. Based on this review, (1) is believed to be novel.

PROOF OF THE IDENTITY. We consider two special cases.

CASE 1 ($j = 0$). Equation (1) reduces to:

$$\sum_{0 \leq l' \leq l} \frac{1}{m-l'} \binom{m+l'}{2l'+1} \binom{m-l'}{2l-2l'} = \frac{1}{2l+1} \binom{2m}{2l}. \quad (8)$$

Divide both sides of (8) by the right-hand side and denote the resulting left-hand side by $S(m, l)$. Then $S(m, l)$ satisfies the recurrence equation $S(m+1, l) - S(m, l) = 0$ —obtained by using Zeilberger's [5] Ekhad, a computer algebra package which is available from <http://www.math.temple.edu/~zeilberg/>—and hence the identity follows from the fact that $S(1, 0) = 1$.

CASE 2 ($j \neq 0$). Equation (1) reduces to

$$\sum_{l'} \sum_{j'} \binom{m+l'-j'}{2l'-j'+1} \binom{m-l'+j'-1}{j-1} \binom{m-l'}{2(l-l')-(j-j')} \binom{j}{j'} = \binom{2m}{2l} \binom{2l}{j-1}, \quad (9)$$

which by multiplying both sides by $(2l-j+1)/j$ is also expressible as

$$\sum_{l'} \sum_{j'} \binom{m-1-l'+j'}{j'} \binom{m-l'}{j-j'} \binom{m+l'-j'}{2l-j} \binom{2l-j+1}{2l'-j'+1} = \binom{2m}{2l} \binom{2l}{j} = \binom{2m}{j, 2l-j}, \quad (10)$$

where

$$\binom{a}{b, c} := \frac{a!}{b!c!(a-b-c)!}. \quad (11)$$

Equation (10) follows from the identity

$$\sum_{l'} \sum_{j'} \binom{p-1-l'+j'}{j'} \binom{p-l'}{j-j'} \binom{m+l'-j'}{k} \binom{k+1}{2l'-j'+m-p+1} = \binom{m+p}{j, k} \quad (12)$$

with $p = m$ and $k = 2l - j$.

Denote the left-hand side of (12) by $S(m, p, j, k)$. $S(m, p, j, k)$ satisfies $S(m+1, p, j, k) = S(m, p, j, k)$ and hence $S(m, p, j, k) = S(m+p, 0, j, k)$. Hence to prove (12) it suffices to prove

$$S(n, 0, j, k) = \binom{n}{j, k} \quad \forall n, j, k \in \mathbb{Z}_{\geq 0}. \quad (13)$$

Clearly (13) is true for $n = 0$. Now, let $n > 0$ and set $S(n, j, k) := S(n, 0, j, k)$. Then $S(n, j, k)$ satisfies the recurrence equation

$$\begin{aligned} & (-1+j-n)S(n-1, j-1, k) - (1+k)S(n-1, j, k-1) \\ & + (j-k-n-1)S(n-1, j, k) + (j+k-n-1)S(n, j-1, k) \\ & + (k+1)S(n, j-1, k+1) + (j+2k-n+1)S(n, j, k) + 2(1+k)S(n, j, k+1) = 0 \end{aligned} \quad (14)$$

that is obtained by using Wegschaider's [4] MultiSum, a computer algebra package which is available from <http://www.risc.uni-linz.ac.at/research/combinat/risc/software/>. Note that the right-hand side of (13) also satisfies (14). Hence by induction it follows that

$$S(n, j, k) = \binom{n}{j, k} \quad \forall n, j, k \in \mathbb{Z}_{\geq 0}. \quad (15) \quad \square$$

REFERENCES

- [1] H. W. Gould, *Combinatorial Identities*. A standardized set of tables listing 500 binomial coefficient summations, Henry W. Gould, Morgantown, W. Va., 1972. [MR 50#6879](#). [Zbl 241.05011](#).
- [2] P. Ribenboim, *13 Lectures on Fermat's Last Theorem*, Springer-Verlag, New York, 1979. [MR 81f:10023](#). [Zbl 456.10006](#).
- [3] J. Riordan, *Combinatorial Identities*, John Wiley & Sons, New York, 1968. [MR 38#53](#). [Zbl 194.00502](#).
- [4] K. Wegschaider, *Computer generated proofs of binomial multi-sum identities*, Diploma thesis, RISC, J. Kepler University, Linz, May 1997.
- [5] D. Zeilberger, *A fast algorithm for proving terminating hypergeometric identities*, Discrete Math. **80** (1990), no. 2, 207–211. [MR 91d:33006](#). [Zbl 701.05001](#).

JOSEPH SINYOR: BELL NEXXIA, SUITE 350, 181 BAY ST., TORONTO, ON, CANADA M5J 2T3
E-mail address: joseph.sinyor@bellnexusia.com

TED SPEEVAK: BELL CANADA, FLOOR 4, 15 ASQUITH AVE, TORONTO, ON, CANADA M4W1J7
E-mail address: ted.speevak@bell.ca

AKALU TEFERA: DEPARTMENT OF MATHEMATICS AND STATISTICS, GRAND VALLEY STATE UNIVERSITY, ALLENDALE, MI 49401, USA
E-mail address: teferaa@gvsu.edu