DEVELOPMENT OF SINGULARITIES IN SOLUTIONS OF A HYPERBOLIC SYSTEM

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ABSTRACT. We consider a special type of a hyperbolic system and show that classical solutions blow up in finite time even for small initial data.

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1. Introduction. For the system of nonlinear elasticity

$$u_t(x, t) = \varphi(v(x, t))v_x(x, t), \quad v_t(x, t) = u_x(x, t), \quad (1.1)$$

it is well known that C^1 -solutions break down in finite time however smooth and small the initial data are. This was shown by Lax [4] in 1964. In his work, the author studied (1.1), for $\varphi > 0$ and $\varphi' > 0$, and established a blowup result. MacCamy and Mizel [7] in 1967 considered the same system and proved a similar result, allowing φ' to change sign. They also showed, under appropriate conditions on φ , that there are x -intervals, for which the solution must exist for all time even though it blows up for values of x outside these intervals.

Messaoudi [9] discussed the following system:

$$u_t(x, t) = \alpha(x)\varphi(v(x, t))v_x(x, t), \quad v_t(x, t) = u_x(x, t), \quad (1.2)$$

which models a transverse motion of a string with variable density. He showed that C^1 -solutions develop singularities in finite time if the initial data are taken with large enough gradients. He also discussed, in [8], a system with dissipation of the form

$$\theta_t + c(\theta)q_x = 0, \quad q_t + \sigma(\theta)\theta_x = -\lambda(\theta)q, \quad (1.3)$$

which describes heat propagation in materials that predict finite propagation speed. This phenomenon is called second sound. Here θ is the difference temperature and q is the heat flux. He studied the Cauchy problem and proved a blowup result of the classical solutions. We should note that, for λ constant and $c(\theta) = -1$, (1.3) reduces to a system describing steady shearing flows in nonlinear viscoelastic fluids. This problem was studied by Slemrod [11] and a blowup result for classical solutions has been established. A similar problem was also discussed by Nishibata [10], Kosiński [3], and Zheng [12] and results concerning global existence and nonexistence have been accomplished.

For more general systems, it is worth mentioning the work of Li et al. [6], in which they discussed

$$u_t(x, t) = A(u(x, t))u_x(x, t), \quad (1.4)$$

associated with decaying initial data. Here $u : I \times (0, T) \rightarrow \mathbb{R}^n$ is a vector-valued function, A is an $(n \times n)$ -matrix, and I is an interval (bounded or unbounded). They proved a global C^1 -solution for the Cauchy problem if, in addition to the local strict hyperbolicity condition, (1.4) is weakly linearly degenerate and the initial data satisfy, for $\mu > 0$, $\sup_x \{(1 + |x|)^{1+\mu} |u'_0(x)| + |u_0(x)|\}$ is small enough. They also established a blowup result to C^1 -solutions for nonweakly linearly degenerate systems. As they pointed out, their work generalizes their result of [5] to the case of initial data with no compact support but they possess certain decay properties.

In this work, we are concerned with a quasilinear hyperbolic system of the form

$$u_t(x, t) = \varphi\left(\frac{v(x, t)}{1 + au(x, t)}\right)v_x(x, t), \quad v_t(x, t) = u_x(x, t), \quad (1.5)$$

where the constant $a \neq 0$. In addition to its importance from the mathematical technique point of view, this system can be regarded as a relative generalization of the one-dimensional wave equation in the sense if $a = 0$, (1.5) reduces to (1.1). We will consider (1.5) together with initial conditions and show that C^1 -solutions blowup even for small initial data. Our result cannot be directly deduced from the results of [6] since we do not impose the same conditions regarding the size and the regularity of the initial data (cf. [6, Theorem 1.2] and Theorem 3.1 below). This work is divided into two parts. In part one we state, without proof, a local existence theorem. In part two our main result is stated and proved.

2. Local existence. We consider the following Cauchy problem

$$u_t(x, t) = \varphi\left(\frac{v(x, t)}{1 + au(x, t)}\right)v_x(x, t), \quad (2.1)$$

$$v_t(x, t) = u_x(x, t), \quad \forall x \in \mathbb{R}, t > 0, \quad (2.2)$$

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad \forall x \in \mathbb{R}, \quad (2.3)$$

where $a \neq 0$ and φ is a function satisfying

$$\varphi(\xi) \geq \beta > 0, \quad \forall \xi \in \mathbb{R}. \quad (2.4)$$

PROPOSITION 2.1. *Assume that φ is a C^1 function satisfying (2.4) and let u_0 and v_0 in $H^2(\mathbb{R})$ be given such that*

$$|1 + au_0(x)| \geq \lambda > 0, \quad \forall x \in \mathbb{R}. \quad (2.5)$$

Then the problem (2.1), (2.2), and (2.3) has a unique local solution (u, v) , on a maximal time interval $[0, T)$, satisfying

$$u, v \in C([0, T), H^2(\mathbb{R})) \cap C^1([0, T), H^1(\mathbb{R})). \quad (2.6)$$

This result can be proved by applying a classical energy argument [1] or the non-linear semigroup theory [2].

REMARK 2.2. The functions u, v are C^1 functions by the standard Sobolev embedding theory.

3. Formation of singularities. We introduce the quantities and the differential operators

$$\begin{aligned} r &:= \frac{1}{a} \ln |1 + au| + \int_0^{v/(1+au)} \alpha(\xi) d\xi, \\ s &:= \frac{1}{a} \ln |1 + au| - \int_0^{v/(1+au)} \beta(\xi) d\xi, \\ \partial_t &:= \frac{\partial}{\partial t} - \rho \left(\frac{v}{1+au} \right) \frac{\partial}{\partial x}, \\ D_t &:= \frac{\partial}{\partial t} + \rho \left(\frac{v}{1+au} \right) \frac{\partial}{\partial x}, \end{aligned} \tag{3.1}$$

where

$$\rho(\xi) = \sqrt{\varphi(\xi)}, \quad \alpha(\xi) = \frac{\sqrt{\varphi(\xi)}}{1 + a\xi\sqrt{\varphi(\xi)}}, \quad \beta(\xi) = \frac{\sqrt{\varphi(\xi)}}{1 - a\xi\sqrt{\varphi(\xi)}}. \tag{3.2}$$

The following lemma shows, for initial data appropriately chosen, that r, s , and ρ are well defined and $|v(x, t)/(1 + au(x, t))|$ is uniformly bounded.

THEOREM 3.1. *Let a and φ be as in Proposition 2.1. Then there exist initial data in $H^2(\mathbb{R})$ satisfying (2.5), for which*

$$\left| \frac{av(x, t)}{1 + au(x, t)} \sqrt{\varphi \left(\frac{v(x, t)}{1 + au(x, t)} \right)} \right| < 1, \quad |1 + au(x, t)| > 0, \tag{3.3}$$

and $|v(x, t)/(1 + au(x, t))|$ is uniformly bounded on $\mathbb{R} \times [0, T)$.

PROOF. We first choose $\delta > 0$ such that if

$$|u_0(x)| < \delta, \quad |v_0(x)| < \delta, \quad \forall x \in \mathbb{R}, \tag{3.4}$$

then

$$\left| \frac{av_0(x)}{1 + au_0(x)} \sqrt{\varphi \left(\frac{v_0(x)}{1 + au_0(x)} \right)} \right| < 1, \quad |1 + au_0(x)| > 0, \quad \forall x \in \mathbb{R}. \tag{3.5}$$

Of course, this is possible by taking δ small enough. Then the continuity of u, v , and φ implies that there exists $T' \leq T$, such that (3.3) holds on $\mathbb{R} \times [0, T')$. Let $T_0 := \sup\{T' : (3.3) \text{ holds for all } x \in \mathbb{R}, t \in [0, T')\}$. We have two cases, either $T_0 = T$, this completes

the proof. Or $T_0 < T$; in this case we estimate

$$\begin{aligned}
 \partial_t r &= \frac{u_t}{1+au} + \alpha \left[\frac{v_t}{1+au} - \frac{v}{(1+au)^2} au_t \right] \\
 &\quad - \rho \left[\frac{u_x}{1+au} + \alpha \frac{v_x}{1+au} - \alpha \frac{v}{(1+au)^2} au_x \right] \\
 &= \frac{1}{1+au} \left[\left(1 - a\alpha \frac{v}{1+au} \right) u_t - \alpha \rho v_x \right] \\
 &\quad + \frac{1}{1+au} \left[\alpha v_t - \rho \left(1 - a\alpha \frac{v}{1+au} \right) u_x \right], \quad \forall x \in \mathbb{R}, t \in [0, T_0].
 \end{aligned} \tag{3.6}$$

We recall that, unless otherwise stated, α , β , ρ , and φ are functions of $v/(1+au)$. By noting that $\alpha\rho = (1 - a\alpha v/(1+au))\varphi$, $(1 - a\alpha - a\alpha v/(1+au))\rho = \alpha$, and using (2.1) and (2.2), we obtain

$$\partial_t r = 0, \quad \forall x \in \mathbb{R}, t \in [0, T_0]. \tag{3.7}$$

Similar calculations also yield

$$D_t s = 0, \quad \forall x \in \mathbb{R}, t \in [0, T_0]. \tag{3.8}$$

Therefore, on $\mathbb{R} \times [0, T_0)$, r and s remain constant along backward and forward characteristics, respectively; hence $\|r\|_\infty = \|r_0\|_\infty$ and $\|s\|_\infty = \|s_0\|_\infty$. It is easy to see that

$$r(x, t) - s(x, t) = \phi \left(\frac{v(x, t)}{1+au(x, t)} \right), \quad \forall x \in \mathbb{R}, t \in [0, T_0], \tag{3.9}$$

where $\phi(\tau) = 2 \int_0^\tau \sqrt{\varphi(\xi)/(1 - a^2 \xi^2 \varphi(\xi))} d\xi$ is strictly monotone and continuous at least in a neighborhood of zero, so it admits a continuous inverse ψ near zero. Since the function $g(\xi) = 1 - a^2 \xi^2 \varphi(\xi)$ is continuous and $g(0) = 1$, one can choose γ so that $g(\xi) \geq \varepsilon > 0$, for all $|\xi| < \gamma$ and choose $\delta_1 > 0$ so that $|\psi(\tau)| < \gamma$, for all $|\tau| < \delta_1$. Therefore, by choosing δ small enough so that (3.4) holds and $\|r_0\|_\infty + \|s_0\|_\infty < \delta_1$, we get

$$|r(x, t) - s(x, t)| \leq \|r_0\|_\infty + \|s_0\|_\infty < \delta_1, \tag{3.10}$$

consequently

$$\left| \frac{v(x, t)}{1+au(x, t)} \right| = |\psi(r-s)| < \gamma, \tag{3.11}$$

which yields

$$\left| \frac{av(x, t)}{1+au(x, t)} \sqrt{\varphi \left(\frac{v(x, t)}{1+au(x, t)} \right)} \right| \leq 1 - \varepsilon < 1, \quad \forall x \in \mathbb{R}, t \in [0, T_0]. \tag{3.12}$$

We then use (3.1), the boundedness of r , and the fact that $1 + a\xi\sqrt{\varphi(\xi)} \geq \varepsilon$ to conclude that $\ln|1+au|$ is bounded on $\mathbb{R} \times [0, T_0]$; hence $|1+au| > 0$. Again by continuity, there

exists $T_1 > T_0$ such that (3.3) holds on $\mathbb{R} \times [0, T_1)$. This contradicts the maximality of T_0 ; hence T_0 must be equal to T . Therefore (3.3) and (3.11) hold. This completes the proof. $\square$

THEOREM 3.2. *Assume that, in addition to (2.4), φ satisfies $\varphi'(0) > 0$. Then there exist initial data u_0, v_0 in $H^2(\mathbb{R})$ satisfying (3.4), for which the solution of the problem (2.1), (2.2), and (2.3) blows up in finite time.*

PROOF. We take an x -partial derivative of (3.7) to get

$$(\partial_t r)_x = r_{xt} - \rho r_{xx} - r_x \rho_x = 0 \quad (3.13)$$

which, in turn, implies

$$\partial_t r_x = r_x \rho_x = \frac{\varphi'}{2\sqrt{\varphi}} r_x \frac{\partial}{\partial x} \left(\frac{v}{1+au} \right). \quad (3.14)$$

We then use

$$r_x = \frac{u_x}{1+au} + \alpha \cdot \frac{\partial}{\partial x} \left(\frac{v}{1+au} \right), \quad s_x = \frac{u_x}{1+au} - \beta \cdot \frac{\partial}{\partial x} \left(\frac{v}{1+au} \right), \quad (3.15)$$

and substitute in (3.14) to arrive at

$$\begin{aligned} \partial_t r_x &= \frac{\varphi'}{2\sqrt{\varphi}(\alpha+\beta)} r_x (r_x - s_x) \\ &= \frac{\varphi'}{4\varphi} \left(1 - a^2 \left(\frac{v}{1+au} \right)^2 \varphi \right) r_x^2 - \frac{\varphi'}{4\varphi} \left(1 - a^2 \left(\frac{v}{1+au} \right)^2 \varphi \right) r_x s_x. \end{aligned} \quad (3.16)$$

To handle the last term in (3.16), we set $W := \varphi^{1/4} r_x$ and substitute in (3.16), to get

$$\begin{aligned} \partial_t W &= \varphi^{1/4} \frac{\varphi'}{4a\varphi} \left(1 - a^2 \left(\frac{v}{1+au} \right)^2 \varphi \right) r_x^2 - \varphi^{1/4} \frac{\varphi'}{4a\varphi} \left(1 - a^2 \left(\frac{v}{1+au} \right)^2 \varphi \right) r_x s_x \\ &\quad + \frac{1}{4} \varphi^{-3/4} \varphi' r_x \partial_t \left(\frac{v}{1+au} \right). \end{aligned} \quad (3.17)$$

By using (2.1) and (2.2), we see that

$$\begin{aligned} \partial_t \left(\frac{v}{1+au} \right) &= \frac{(1+au)(v_t - \sqrt{\varphi} v_x) - av(u_t - \sqrt{\varphi} u_x)}{(1+au)^2} \\ &= \frac{(1+au)(u_x - \sqrt{\varphi} v_x) - av(\varphi v_x - \sqrt{\varphi} u_x)}{(1+au)^2} \\ &= \frac{(u_x - \sqrt{\varphi} v_x)(1+au + a\sqrt{\varphi} v)}{(1+au)^2}. \end{aligned} \quad (3.18)$$

Also straightforward computations lead to

$$s_x = \frac{1}{\sqrt{\varphi}} \frac{\beta}{1+au} (u_x - \sqrt{\varphi} v_x) = \frac{(u_x - \sqrt{\varphi} v_x)}{1+au - av\sqrt{\varphi}}. \quad (3.19)$$

By combining (3.17), (3.18), and (3.19), we arrive at

$$\partial_t W = \varphi^{-5/4} \frac{\varphi'}{4} \left(1 - a^2 \left(\frac{v}{1+au} \right)^2 \varphi \right) W^2. \quad (3.20)$$

If we choose δ sufficiently small, the coefficient of the quadratic term in (3.20) remains bounded away from zero; that is, $\varphi^{-5/4} \varphi' (1 - a^2 (v/(1+au))^2 \varphi) / 4 \geq k > 0$. Consequently, (3.20) gives

$$\partial_t W \geq kW^2. \quad (3.21)$$

Therefore, by choosing initial data small enough and satisfying (3.4) with derivatives such that $W_0 > 0$, (3.21) shows that W (hence r_x) blows up in finite time. This completes the proof. $\square$

REMARK 3.3. Similar result can be obtained for $\varphi'(0) < 0$. In this case consider the evolution of s_x on the forward characteristics.

REMARK 3.4. A simple integration of (3.21) shows that the larger W_0 is, the quicker the blowup takes place.

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