DIMENSIONS OF PRYM VARIETIES

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ABSTRACT. Given a tame Galois branched cover of curves $\pi : X \rightarrow Y$ with any finite Galois group G whose representations are rational, we compute the dimension of the (generalized) Prym variety $\text{Prym}_\rho(X)$ corresponding to any irreducible representation ρ of G . This formula can be applied to the study of algebraic integrable systems using Lax pairs, in particular systems associated with Seiberg-Witten theory. However, the formula is much more general and its computation and proof are entirely algebraic.

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1. Introduction. The most familiar Prym variety arises from a (possibly branched) double cover $\pi : X \rightarrow Y$ of curves. In this situation, there is a surjective norm map $\text{Nm} : \text{Jac}(X) \rightarrow \text{Jac}(Y)$, and the Prym (another Abelian variety) is a connected component of its kernel. Another way to think of this is that the involution σ of the double cover induces an action of $\mathbb{Z}/2\mathbb{Z}$ on the vector space $H^0(X, \omega_X)$, which can then be decomposed as a representation of $\mathbb{Z}/2\mathbb{Z}$. The Jacobian of the base curve Y and the Prym correspond to the trivial and sign representations, respectively. The Prym variety can be defined as the component containing the identity of $(\text{Jac}(X) \otimes_{\mathbb{Z}} \varepsilon)^\sigma$, where ε denotes the sign representation of $\mathbb{Z}/2\mathbb{Z}$.

The generalization of this construction that we study in this paper is as follows. Let G be a finite group, and $\pi : X \rightarrow Y$ be a tame Galois branched cover, with Galois group G , of smooth projective curves over an algebraically closed field. The action of G on X induces an action on the vector space of differentials $H^0(X, \omega_X)$, and on the Jacobian $\text{Jac}(X)$. For any representation ρ of G , we define $\text{Prym}_\rho(X)$ to be the connected component containing the identity of $(\text{Jac}(X) \otimes_{\mathbb{Z}} \rho^*)^G$. The vector space $H^0(X, \omega_X)$ decomposes as a $\mathbb{Z}[G]$ -module into a direct sum of isotypic pieces

$$H^0(X, \omega_X) = \bigoplus_{j=1}^N \rho_j \otimes V_j, \quad (1.1)$$

where $\rho_1, \dots, \rho_N$ are the irreducible representations of G . If G is such that all of its representations are rational, then the Jacobian also decomposes, up to isogeny, into a direct sum of Pryms [5]:

$$\text{Jac}(X) \sim \bigoplus_{j=1}^N \rho_j \otimes \text{Prym}_{\rho_j}(X). \quad (1.2)$$

In particular, if G is the Weyl group of a semisimple Lie algebra, then it satisfies this property.

The goal of this paper is to compute the dimension of such a Prym variety. This formula is given in [Section 2](#), with a proof that uses only the Riemann-Hurwitz theorem and some character theory. Special cases of this formula relevant to integrable systems have appeared previously [[2](#), [11](#), [12](#), [13](#)].

One motivation for this work comes from the study of algebraically integrable systems. An algebraically integrable system is a Hamiltonian system of ordinary differential equations, where the phase space is an algebraic variety with an algebraic (holomorphic, over $\mathbb{C}$) symplectic structure. The complete integrability of the system means that there are n commuting Hamiltonian functions on the $2n$ -dimensional phase space. For an algebraically integrable system, these functions should be algebraic, in which case they define a morphism to an n -dimensional space of states for the system. The flow of the system will be linearized on the fibers of this morphism, which, if they are compact, will be n -dimensional Abelian varieties.

Many such systems can be solved by expressing the system as a Lax pair depending on a parameter z . The equations can be written in the form $(d/dt)A = [A, B]$, where A and B are elements of a Lie algebra $\mathfrak{g}$, and depend both on time t and on a parameter z , which is thought of as a coordinate on a curve Y . In this case, the flow of the system is linearized on a subtorus of the Jacobian of a Galois cover of Y . If it can be shown that this subtorus is isogenous to a Prym of the correct dimension, then the system is completely integrable.

In [Section 3](#), we briefly discuss two examples of such systems, the periodic Toda lattice and Hitchin systems. Both of these are important in Seiberg-Witten theory, providing solutions to $\mathcal{N} = 2$ supersymmetric Yang-Mills gauge theory in four dimensions.

2. Dimensions. We can start by using the Riemann-Hurwitz formula to find the genus g_X of X , which will be the dimension of the whole space $H^0(X, \omega_X)$ and of $\text{Jac}(X)$. Since $\pi : X \rightarrow Y$ is a cover of degree $|G|$, we get

$$g_X = 1 + |G|(g - 1) + \frac{\deg R}{2}, \quad (2.1)$$

where g is the genus of the base curve Y and R is the ramification divisor.

The first isotypic piece whose dimension we can find is V_1 , corresponding to the trivial representation. The subspace where G acts trivially is the subspace of differentials which are pullbacks by π of differentials on Y . This tells us that $\dim V_1 = \dim H^0(Y, \omega_Y) = g$.

In the case of classical Pryms, where $G = \mathbb{Z}/2$, there is only one other isotypic piece, V_ε corresponding to the sign representation ε . Thus we have

$$\dim V_\varepsilon = g_X - g = g - 1 + \frac{\deg R}{2}. \quad (2.2)$$

For larger groups G , there are more isotypic pieces, but we also have more information: we can look at intermediate curves, that is, quotients of X by subgroups H of G . Differentials on X/H pull back to differentials on X , where H acts trivially. Thus

$$H^0(X/H, \omega_{X/H}) = \bigoplus_{j=1}^N (\rho_j)^H \otimes V_j. \quad (2.3)$$

The map $\pi_H : X/H \rightarrow Y$ will be a cover of degree $|G|/|H|$, so Riemann-Hurwitz gives us the following formula for the genus g_H of X/H , which is the dimension of $H^0(X/H, \omega_{X/H})$:

$$g_H = 1 + \frac{|G|}{|H|} (g - 1) + \frac{\deg R_H}{2}, \quad (2.4)$$

where again R_H is the ramification divisor.

We can further analyze the ramification divisor, by classifying the branch points according to their inertial groups. Since $\pi : X \rightarrow Y$ is a Galois cover of curves over $\mathbb{C}$, all of the inertial groups must be cyclic.

LEMMA 2.1. *Let G be a finite group all of whose characters are defined over $\mathbb{Q}$. If two elements $x, y \in G$ generate conjugate cyclic subgroups, then they are conjugate.*

PROOF (adapted from [3]). We want to show that for any character χ of G , $\chi(x) = \chi(y)$. Then the properties of characters will tell us that x and y must be in the same conjugacy class.

We may assume that x and y generate the same subgroup H . Then $y = x^k$ for some integer k relatively prime to $|H|$. Let χ be a character of G , and $\rho : G \rightarrow \mathrm{GL}(n, \mathbb{C})$ a representation with character χ . Then $\rho(x)$ will be a matrix with eigenvalues $\lambda_1, \dots, \lambda_n$, and $\rho(y)$ will have eigenvalues $\lambda_1^k, \dots, \lambda_n^k$. Since $x^{|H|} = 1$, we have $\lambda_1^{|H|} = \dots = \lambda_n^{|H|} = 1$. Let ξ be a primitive $|H|$ th root of unity. Then we can write $\lambda_1 = \xi^{v_1}, \dots, \lambda_n = \xi^{v_n}$ for some integers v_i . Now $\chi(x) = \mathrm{Trace}(\rho(x)) = \lambda_1 + \dots + \lambda_n$, and $\chi(y) = \chi(x^k) = \lambda_1^k + \dots + \lambda_n^k$. Thus $\chi(y)$ will be the image of $\chi(x)$ under the element of $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ which sends $\xi \mapsto \xi^k$. Since the values of χ are rational, this element will act trivially, so $\chi(y) = \chi(x)$. $\square$

From now on, we suppose that G is such that all of its characters are rational. (This is true, for instance, if G is a Weyl group.) Pick representative elements $h_1, \dots, h_N$ for each conjugacy class in G , and let $H_1, \dots, H_N$ be the cyclic groups that each of them generates. By Lemma 2.1, this is the whole set (up to conjugacy) of cyclic subgroups of G . We can partially order this set of cyclic subgroups by their size, so that H_1 is the trivial subgroup. Now we can classify the branch points: let R_k , $k = 2, \dots, N$ be the degree of the branch locus with inertial group conjugate to H_k (ignoring the trivial group). Over each point of the branch locus, where the inertial group is conjugate to H_k , there are $|G|/|H_k|$ points in the fiber. Thus the degree of the ramification divisor R of $\pi : X \rightarrow Y$ is

$$\deg R = \sum_{k=1}^N \left(|G| - \frac{|G|}{|H_k|} \right) R_k. \quad (2.5)$$

For each quotient curve X/H , each point in the fiber of $\pi_H : X/H \rightarrow Y$ over a point with inertial group H_k corresponds to a double coset $H_k \backslash G/H$. Thus the degree of the ramification divisor R_H is

$$\deg R_H = \sum_{k=1}^N \left(\frac{|G|}{|H|} - \#(H_k \backslash G/H) \right) R_k. \quad (2.6)$$

Combining (2.5) and (2.6) with the earlier Riemann-Hurwitz computations ((2.1) and (2.4)), we get

$$\begin{aligned} g_X &= 1 + |G|(\mathfrak{g} - 1) + \sum_k \left(|G| - \frac{|G|}{|H|} \right) \frac{R_k}{2}, \\ g_H &= 1 + \frac{|G|}{|H|}(\mathfrak{g} - 1) + \sum_k \left(\frac{|G|}{|H|} - \#(H_k \backslash G/H) \right) \frac{R_k}{2}. \end{aligned} \quad (2.7)$$

Since the genera g_H are exactly the dimensions $\dim H^0(X/H, \omega_{X/H})$, we also have

$$g_H = \sum_{j=1}^N \dim \rho_j^H \dim V_j. \quad (2.8)$$

For each subgroup H , this is a linear equation for the unknown dimensions $\dim V_j$ in terms of the genus g_H . Thus by taking quotients by the set of all cyclic subgroups $H_1 \cdots H_N$, we get a system of N equations. We wish to invert the matrix $\dim \rho_j^{H_i}$ and find the N unknowns $\dim V_j$.

LEMMA 2.2. *The matrix $\dim \rho_j^{H_i}$ is invertible.*

PROOF. We show that the rows of the matrix are linearly independent, using the fact that rows of the character table are linearly independent. First, note that $\dim \rho_j^{H_i}$, the dimension of the subspace of ρ_j invariant under H_i , is equal to the inner product of characters $\langle \text{Res}_{H_i}^G \rho_j, 1 \rangle$, which we can read off from the character table of G as

$$\dim \rho_j^{H_i} = \frac{1}{|H_i|} \sum_{a_i \in H_i} \chi_{\rho_j}(a_i). \quad (2.9)$$

Compare this matrix to the matrix of the character table $\chi_{\rho_j}(a_i)$. From (2.9) we see that each row is a sum of multiples of rows of the character table. Since each element of a subgroup has order less than or equal to the order of the subgroup, the rows of the character table being added to get row i appear at or below row i in the character table. Thus if we write the matrix $\dim \rho_j^{H_i}$ in terms of the basis of the character table, we get a lower triangular matrix with nonzero entries on the diagonal. By row reduction, we see that the linear independence of the rows of $\dim \rho_j^{H_i}$ is equivalent to the linear independence of the rows of the character table. $\square$

THEOREM 2.3. *For each nontrivial irreducible representation ρ_j of G , V_j has dimension*

$$(\dim \rho_j)(\mathfrak{g} - 1) + \sum_{k=1}^N \left((\dim \rho_j) - (\dim \rho_j^{H_k}) \right) \frac{R_{H_k}}{2}. \quad (2.10)$$

PROOF. Since the matrix $\dim \rho_j^{H_i}$ is invertible, there is a unique solution to the system of (2.8), so we only need to show that this is a solution. Namely, given this formula for $\dim V_j$ and combining (2.7) and (2.8), we wish to show that for each cyclic subgroup H_i ,

$$\sum_{j=1}^N \dim \rho_j^{H_i} \dim V_j = 1 + \frac{|G|}{|H_i|}(\mathfrak{g} - 1) + \sum_k \left(\frac{|G|}{|H_i|} - \#(H_k \backslash G/H_i) \right) \frac{R_k}{2}. \quad (2.11)$$

Note that on the left-hand side we are summing over all representations, not just the nontrivial ones, so our notation will be simpler if we write $\dim V_1 = g$ in a similar form to (2.8). For the trivial representation ρ_1 , $(\dim \rho_1) - (\dim \rho_1^{H_k}) = 0$ (since ρ_1 is fixed by any subgroup H_k), so

$$\dim V_1 = 1 + (\dim \rho_1)(g-1) + \sum_{k=1}^N \left((\dim \rho_1) - (\dim \rho_1^{H_k}) \right) \frac{R_{H_k}}{2}. \quad (2.12)$$

The sum on the left-hand side of (2.11) will be

$$1 + \sum_{j=1}^N \dim \rho_j^{H_i} \left((\dim \rho_j)(g-1) + \sum_{k=1}^N \left((\dim \rho_j) - (\dim \rho_j^{H_k}) \right) \frac{R_{H_k}}{2} \right). \quad (2.13)$$

We look at the $(g-1)$ term and the R_{H_k} terms separately. For the $(g-1)$ coefficient, we can write both $\dim \rho_j^{H_i}$, and $\dim \rho_j$ in terms of characters of G (as in (2.9)) and exchange the order of summation to get

$$\sum_{j=1}^N \dim \rho_j^{H_i} \dim \rho_j = \frac{1}{|H_i|} \sum_{a_i \in H_i} \sum_{j=1}^N \chi_{\rho_j}(a_i) \chi_{\rho_j}(e), \quad (2.14)$$

where e is the identity element of G . The inner sum amounts to take the inner product of two columns of the character table of G . The orthogonality of characters tells us that this inner product will be zero unless the two columns are the same, in this case if $a_i = e$. Thus the sum over elements in H_i disappears, and we get the sum of the squares of the dimensions of the characters

$$\frac{1}{|H_i|} \sum_{j=1}^N \chi_{\rho_j}(e)^2 = \frac{|G|}{|H_i|}, \quad (2.15)$$

which is what we want.

The R_{H_k} term looks like

$$\sum_{j=1}^N \dim \rho_j^{H_i} \sum_{k=1}^N \left((\dim \rho_j) - (\dim \rho_j^{H_k}) \right) \frac{R_{H_k}}{2}. \quad (2.16)$$

We can distribute and rearrange the sums to get

$$\sum_{k=1}^N \left(\sum_{j=1}^N \dim \rho_j^{H_i} \dim \rho_j - \sum_{j=1}^N \dim \rho_j^{H_i} \dim \rho_j^{H_k} \right) \frac{R_{H_k}}{2}. \quad (2.17)$$

As in (2.14) and (2.15), the first term becomes $|G|/|H_i|$. The second term is also the inner product of columns of the character table:

$$\sum_{j=1}^N \dim \rho_j^{H_i} \dim \rho_j^{H_k} = \frac{1}{|H_i|} \frac{1}{|H_k|} \sum_{a_i \in H_i} \sum_{a_k \in H_k} \sum_{j=1}^N \chi_{\rho_j}(a_i) \chi_{\rho_j}(a_k). \quad (2.18)$$

This will be zero unless a_i and a_k are conjugate, in which case $\chi_{\rho_j}(a_i) = \chi_{\rho_j}(a_k)$ and character theory tells us (cf. [8, page 18]) that

$$\sum_{j=1}^N \chi_{\rho_j}(a_i)^2 = \frac{|G|}{c(a_i)}, \quad (2.19)$$

where $c(a_i)$ is the number of elements in the conjugacy class of a_i . Now the second term has become

$$\frac{|G|}{|H_i| |H_k|} \sum_{\{a_i, a_k\}} \frac{1}{c(a_i)}, \quad (2.20)$$

where the sum is taken over pairs of elements $a_i \in H_i$, $a_k \in H_k$ such that a_i and a_k are conjugate. This is exactly the number of double cosets $\#(H_k \backslash G / H_i)$.

Adding up all of the terms, the sum on the left-hand side becomes

$$1 + \frac{|G|}{|H_i|} (g-1) + \left(\frac{|G|}{|H_i|} - \#(H_k \backslash G / H_i) \right) \frac{R_{H_k}}{2}, \quad (2.21)$$

which is exactly the right-hand side. $\square$

COROLLARY 2.4. *For each nontrivial irreducible representation ρ_j of G , $\text{Prym}_{\rho_j}(X)$ has dimension*

$$(\dim \rho_j)(g-1) + \sum_{k=1}^N \left((\dim \rho_j) - (\dim \rho_j^{H_k}) \right) \frac{R_{H_k}}{2}. \quad (2.22)$$

3. Integrable systems

3.1. Periodic Toda lattice. The periodic Toda system is a Hamiltonian system of differential equations with Hamiltonian

$$H(p, q) = \frac{|p|^2}{2} + \sum_{\alpha} e^{\alpha(q)}, \quad (3.1)$$

where p and q are elements of the Cartan subalgebra $\mathfrak{t}$ of a semisimple Lie algebra $\mathfrak{g}$, and the sum is over the simple roots of $\mathfrak{g}$ plus the highest root. This system can be expressed in Lax form [1] $(d/dt)A = [A, B]$, where A and B are elements of the loop algebra $\mathfrak{g}^{(1)}$, and can be thought of as elements of $\mathfrak{g}$ which depend on a parameter $z \in \mathbb{P}^1$. For $\mathfrak{sl}(n)$, A is of the form

$$\begin{pmatrix} \gamma_1 & 1 & & x_0 z \\ x_1 & \gamma_2 & \ddots & \\ & \ddots & \ddots & 1 \\ z & & x_{n-1} & \gamma_n \end{pmatrix}. \quad (3.2)$$

For any representation ϱ of $\mathfrak{g}$, the spectral curve S_{ϱ} defined by the equation $\det \varrho(A(z) - \lambda I) = 0$ is independent of time (i.e., is a conserved quantity of the system). The spectral curve is a finite cover of $\mathbb{P}^1$ which for generic z parameterizes the

eigenvalues of $\varrho(A(z))$. While the eigenvalues are conserved by the system, the eigenvectors are not. The eigenvectors of $\varrho(A)$ determine a line bundle on the spectral cover, so an element of $\text{Jac}(S_\varrho)$. The flow of the system is linearized on this Jacobean. Since the original system of equations did not depend on a choice of representation ϱ , the flow is actually linearized on an Abelian variety which is a subvariety of $\text{Jac}(S_\varrho)$ for every ϱ .

In fact, instead of considering each spectral cover we can look at the cameral cover $X \rightarrow \mathbb{P}^1$. This is constructed as a pullback to $\mathbb{P}^1$ of the cover $\mathfrak{t} \rightarrow \mathfrak{t}/G$, where G is the Weyl group of $\mathfrak{g}$. This cover is pulled back by the rational map $\mathbb{P}^1 \rightarrow \mathfrak{t}/G$ defined by the class of $A(z)$ under the adjoint action of the corresponding Lie group. (For $A(z)$ a regular semisimple element of $\mathfrak{sl}(n)$, this map sends z to the unordered set of eigenvalues of $A(z)$.) Thus, the cameral cover is a finite Galois cover of $\mathbb{P}^1$ whose Galois group G is the Weyl group of $\mathfrak{g}$. The flow of the Toda system is linearized on the Prym of this cover corresponding to the representation of G on $\mathfrak{t}^*$. This is an r -dimensional representation, where r is the rank, so the dimension of this Prym is

$$r(-1) + \sum_{k=1}^N (r - (\dim \mathfrak{t}^{H_k})) \frac{R_{H_k}}{2}. \quad (3.3)$$

The ramification of this cover has been analyzed in [6, 11]. There are $2r$ branch points where the inertial group H is $\mathbb{Z}/2\mathbb{Z}$ generated by one reflection, so for each of these $\dim \mathfrak{t}^H$ is $r-1$. There are also two points ($z=0$ and ∞) where the inertial group H is generated by the Coxeter element, the product of the reflections corresponding to the simple roots. This element of G does not fix any element of $\mathfrak{t}$, so for these two points $\dim \mathfrak{t}^H = 0$. Thus the dimension of the Prym is

$$-r + (r - (r-1)) \frac{2r}{2} + (r-0) \frac{2}{2} = r. \quad (3.4)$$

Since the original system of equations had a $2r$ -dimensional phase space, this is the answer that we want.

3.2. Hitchin systems. Hitchin [9] showed that the cotangent bundle to the moduli space of semistable vector bundles on a curve Y has the structure of an algebraically completely integrable system. His proof, later extended to principal $\mathcal{G}$ bundles with any reductive Lie group $\mathcal{G}$ [7, 13], uses the fact that this moduli space is equivalent (by deformation theory) to the space of *Higgs pairs*, pairs (P, ϕ) of a principal bundle, and an endomorphism $\phi \in H^0(Y, \text{ad}(P) \otimes \omega_Y)$. As in the case of the Toda system, the key construction is of a cameral cover of Y . The eigenvalues of ϕ , which are sections of the line bundle ω_Y , determine a spectral cover of Y in the total space of the bundle. The eigenvectors determine a line bundle on this spectral cover. The Hitchin map sends a Higgs pair (P, ϕ) to the set of coefficients of the characteristic polynomial. Each coefficient is a section of a power of ω_Y , so the image of the Hitchin map is $B := \bigoplus_{i=1}^r H^0(Y, \omega_Y^{\otimes d_i})$, where the d_i are the degrees of the basic invariant polynomials of the Lie algebra $\mathfrak{g}$.

Again, we can consider instead the cameral cover $X_b \rightarrow Y$, which is obtained as a pullback to Y via ϕ of $\mathfrak{t} \otimes \omega_Y \rightarrow \mathfrak{t} \otimes \omega_Y/G$. The generic fiber of the Hitchin map is

isogenous to $\mathrm{Prym}_{\mathfrak{t}}(X)$, which has dimension

$$r(g-1) + \sum_{k=1}^N (r - (\dim \mathfrak{t}^{H_k})) \frac{R_{H_k}}{2}. \quad (3.5)$$

By looking at the generic fiber, we can restrict our attention to cameral covers where the only ramification is of order two, with inertial group H generated by one reflection. The last piece of information we need to compute the dimension is the degree of the branch divisor of $X \rightarrow Y$.

The cover $\mathfrak{t} \otimes \omega_Y \rightarrow \mathfrak{t} \otimes \omega_Y / G$ is ramified where any of the roots, or their product, is equal to zero. There are $(\dim \mathfrak{g} - r)$ roots, so this defines a hypersurface of degree $(\dim \mathfrak{g} - r)$ in the total space of ω_Y . The ramification divisor of $X \rightarrow Y$ is the intersection of this hypersurface with the section ϕ , which is the divisor corresponding to the line bundle $\omega_Y^{\otimes (\dim \mathfrak{g} - r)}$. Thus the degree of the branch divisor will be $(\dim \mathfrak{g} - r)(2g - 2)$.

Combining all of this information, we see that the dimension of the Prym is

$$\begin{aligned} \dim \mathrm{Prym}_{\mathfrak{t}}(X) &= r(g-1) + (r - (r-1)) \frac{(\dim \mathfrak{g} - r)(2g-2)}{2} \\ &= r(g-1) + (\dim \mathfrak{g} - r)(g-1) \\ &= \dim \mathfrak{g}(g-1). \end{aligned} \quad (3.6)$$

By comparison, the dimension of the base space is

$$\sum_{i=1}^r h^0(Y, \omega_Y^{d_i}). \quad (3.7)$$

The sum of the degrees d_i of the basic invariant polynomials of $\mathfrak{g}$ is the dimension of a Borel subalgebra, $(\dim \mathfrak{g} + r)/2$. For $g > 1$, Riemann-Roch gives

$$\sum_{i=1}^r h^0(Y, \omega_Y^{d_i}) = \sum_{i=1}^r (2d_i - 1)(g-1) = (\dim \mathfrak{g} + r - r)(g-1) = \dim \mathfrak{g}(g-1). \quad (3.8)$$

Which, as Hitchin said, “somewhat miraculously” turns out to be the same thing.

Markman [10] and Bottacin [4] generalized the Hitchin system by twisting the line bundle ω_Y by an effective divisor D . The effect of this is to create a family of integrable systems, parameterized by the residue of the Higgs field ϕ at D . The base space of each system is a fiber of the map

$$\begin{array}{c} B := \bigoplus_{i=1}^r H^0(Y, \omega_Y(D)^{\otimes d_i}) \\ \downarrow \\ \bar{B} := \text{the space of possible residues at } D, \end{array} \quad (3.9)$$

which sends the set of r sections in B to its set of residues at D . At each point of D , there are r independent coefficients, so the dimension of $\bar{B}$ is $r(\deg D)$. Thus the base

space of each system has dimension

$$\begin{aligned}
 \dim B - \dim \tilde{B} &= \sum_{i=1}^r h^0(Y, \omega_Y(D)^{\otimes d_i}) - r(\deg D) \\
 &= \sum_{i=1}^r (d_i(2g - 2 + \deg D) - (g - 1)) - r(\deg D) \\
 &= \frac{1}{2}(\dim \mathcal{G} + r)(2g - 2 + \deg D) - r(g - 1) - r(\deg D) \\
 &= (\dim \mathcal{G})(g - 1) + \frac{\dim \mathcal{G} - r}{2} \deg D.
 \end{aligned} \tag{3.10}$$

Markman [10] showed that the generic fiber of this system is again isogenous to $\mathrm{Prym}_t(X)$, where X is a cameral cover of the base curve Y . The construction of the cameral cover is similar to the case of the Hitchin system, except that ϕ is a section of $\mathrm{ad}(P) \otimes \omega_Y(D)$. Thus the ramification divisor is $(\omega_Y(D))^{\otimes (\dim \mathcal{G} - r)}$, and the dimension is

$$\begin{aligned}
 \dim \mathrm{Prym}_t(X) &= r(g - 1) + \frac{(\dim \mathcal{G} - r)(2g - 2 + \deg D)}{2} \\
 &= \dim \mathcal{G}(g - 1) + \frac{(\dim \mathcal{G} - r)}{2} \deg D.
 \end{aligned} \tag{3.11}$$

Again, this is the same dimension as the base of the system.

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