ANALOGUES OF SOME TAUBERIAN THEOREMS FOR STRETCHINGS

RICHARD F. PATTERSON

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Abstract. We investigate the effect of four-dimensional matrix transformation on new classes of double sequences. Stretchings of a double sequence is defined, and this definition is used to present a four-dimensional analogue of D. Dawson’s copy theorem for stretching of a double sequence. In addition, the multidimensional analogue of D. Dawson’s copy theorem is used to characterize convergent double sequences using stretchings.

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1. Introduction. In this paper, RH-regular matrices and the stretching of double sequences are used to characterize $P$-convergent sequences. To achieve this goal we begin by defining an $\epsilon$-Pringsheim-copy and a stretching of double sequences. In addition, the copy theorem of Dawson in [1] will be extended as follows: if each of $A$ and $T$ is an $RH$-regular matrix, and $x$ is any bounded double complex sequence with $\epsilon$ being any bounded positive term double sequence with $P\lim_{i,j} {\epsilon}_{i,j} = 0$, then there exists a stretching $y$ of $x$ such that $T(Ay)$ exists and contains an $\epsilon$-Pringsheim-copy of $x$. By using this extended copy theorem some natural implications and variations of this extended copy theorem will be presented.

2. Definitions, notations, and preliminary results

DEFINITION 2.1 (see [3]). A double sequence $x = [x_{k,l}]$ has Pringsheim limit $L$ (denoted by $P\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k,l > N$. We will describe such an $x$ more briefly as “$P$-convergent.”

DEFINITION 2.2 (see [3]). A double sequence $x$ is called definite divergent, if for every (arbitrarily large) $G > 0$ there exist two natural numbers $n_1$ and $n_2$ such that $|x_{n,k}| > G$ for $n \geq n_1, k \geq n_2$.

DEFINITION 2.3. The double sequence $[y]$ is a double subsequence of the sequence $[x]$ provided that there exist two increasing double index sequences $\{n_j\}$ and $\{k_j\}$ such that if $z_j = x_{n_j,k_j}$, then $y$ is formed by

$$
\begin{array}{cccc}
  z_1 & z_2 & z_3 & z_5 \\
  z_4 & z_3 & z_6 & z_7 \\
  z_9 & z_8 & z_7 & z_{10} \\
 \end{array}
$$

(2.1)
The double sequence $x$ is bounded if and only if there exists a positive number $M$ such that $|x_{k,l}| < M$ for all $k$ and $l$. A two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The Silverman-Toeplitz theorem [5, 6] characterizes the regularity of two-dimensional matrix transformations. In [4], Robison presented a four-dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is $P$-convergent is not necessarily bounded. The definition of regularity for four-dimensional matrices will be stated below along with the Robison-Hamilton characterization of the regularity of four-dimensional matrices.

**Definition 2.4.** The four-dimensional matrix $A$ is said to be $RH$-regular if it maps every bounded $P$-convergent sequence into a $P$-convergent sequence with the same $P$-limit.

**Theorem 2.5** (see [2, 4]). The four-dimensional matrix $A$ is $RH$-regular if and only if

- $(RH_1)$ $P \lim_{m,n} a_{m,n,k,l} = 0$ for each $k$ and $l$;
- $(RH_2)$ $P \lim_{m,n} \sum_{k,l=1}^{\infty} a_{m,n,k,l} = 1$;
- $(RH_3)$ $P \lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0$ for each $l$;
- $(RH_4)$ $P \lim_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0$ for each $k$;
- $(RH_5)$ $\sum_{k,l=1}^{\infty} |a_{m,n,k,l}|$ is $P$-convergent; and
- $(RH_6)$ there exist finite positive integers $A$ and $B$ such that $\sum_{k,l=B}^{\infty} |a_{m,n,k,l}| < A$.

**Example 2.6.** The sequences $[y_{n,k}] = 1$ and $[y_{n,k}] = -1$ for each $n$ and $k$ are both subsequences of the double sequence whose $n,k$th term is $x_{n,k} = (-1)^n$. In addition to the two subsequences given, every double sequence of 1’s and -1’s is a subsequence of this $x$.

**Example 2.7.** As another example of a subsequence of a double sequence, we define $x$ as follows:

$$
\begin{align*}
    x_{n,k} := \begin{cases} 
        1, & \text{if } n = k, \\
        \frac{1}{n}, & \text{if } n < k, \\
        n, & \text{if } n > k.
    \end{cases}
\end{align*}
$$

(2.2)

Then the double sequence

$$
\begin{align*}
    y_{n,k} := \begin{pmatrix}
        \frac{1}{2} & 4 & \frac{1}{10} & 20 & \cdots \\
        18 & 8 & 6 & \frac{1}{12} & 22 & \cdots \\
        32 & 30 & 28 & 26 & \cdots \\
        \cdots & \cdots & \cdots & \cdots & \cdots
    \end{pmatrix}
\end{align*}
$$

(2.3)

is clearly a subsequence of $x$. 
**Remark 2.8.** Note that if the double sequence \( x \) contains at most a finite number of unbounded rows and/or columns, then every subsequence of \( x \) is bounded. In addition, the finite number of unbounded rows and/or columns does not affect the \( P \)-convergence or \( P \)-divergence of \( x \) and its subsequences.

**Definition 2.9.** A number \( \beta \) is called a Pringsheim limit point of the double sequence \( x = [x_{n,k}] \) provided that there exists a subsequence \( y = [y_{n,k}] \) of \( [x_{n,k}] \) that has Pringsheim limit \( \beta : \lim_{n,k} y_{n,k} = \beta \).

**Example 2.10.** Define the double sequence \( x \) by

\[
x_{n,k} := \begin{cases} (-1)^n, & \text{if } n = k, \\ (-2)^n, & \text{if } n = k + 1, \\ 0, & \text{otherwise}. \end{cases} \tag{2.4}
\]

This double sequence has five Pringsheim limit points, namely \(-2, -1, 0, 1, \) and \( 2 \).

**Remark 2.11.** The definition of a Pringsheim limit point can also be stated as follows: \( \beta \) is a Pringsheim limit point of \( x \) provided that there exist two increasing index sequences \( \{n_i\} \) and \( \{k_i\} \) such that \( \lim_{i} x_{n_i,k_i} = \beta \).

**Definition 2.12.** A double sequence \( x \) is divergent in the Pringsheim sense (\( P \)-divergent) provided that \( x \) does not converge in the Pringsheim sense (\( P \)-convergent).

**Remark 2.13.** Definition 2.12 can also be stated as follows: a double sequence \( x \) is \( P \)-divergent provided that either \( x \) contains at least two subsequences with distinct finite Pringsheim limit points or \( x \) contains an unbounded subsequence. Also note that, if \( x \) contains an unbounded subsequence then \( x \) also contains a definite divergent subsequence.

**Example 2.14.** This is an example of a convergent double sequence whose terms form an unbounded set

\[
x_{n,k} := \begin{cases} k, & \text{if } n = 1, \\ n, & \text{if } k = 2, \\ 0, & \text{otherwise}. \end{cases} \tag{2.5}
\]

**Example 2.15.** This is an example of an unbounded divergent double sequence with three finite Pringsheim limit points, namely \(-1, 0, \) and \( 1 \):

\[
x_{n,k} := \begin{cases} k + 1, & \text{if } n = 1, \\ (-1)^{n+1}, & \text{if } n = k, \\ 0, & \text{otherwise}. \end{cases} \tag{2.6}
\]

**Example 2.16.** This is an example of a double sequence which contains an unbounded subsequence

\[
x_{n,k} := \begin{cases} n, & \text{if } n = k, \\ -n, & \text{if } n = k + 1, \\ 0, & \text{otherwise}. \end{cases} \tag{2.7}
\]
**Example 2.17.** For an example of a definite divergent sequence take $x_{n,k} = n$ for each $n$ and $k$; then it is also clear that $x$ contains an unbounded subsequence.

The following propositions are easily verified.

**Proposition 2.18.** If $x = [x_{n,k}]$ is $P$-convergent to $L$ then $x$ cannot converge to a limit $M$, where $M \neq L$.

**Proposition 2.19.** If $x = [x_{n,k}]$ is $P$-convergent to $L$, then any subsequence of $x$ is also $P$-convergent to $L$.

**Remark 2.20.** For an ordinary single-dimensional sequence, any sequence is a subsequence of itself. This, however, is not the case in the two-dimensional plane, as illustrated by the following example.

**Example 2.21.** The sequence

$$x_{n,k} := \begin{cases} 
1, & \text{if } n = k = 0, \\
1, & \text{if } n = 0, \ k = 1, \\
1, & \text{if } n = 1, \ k = 0, \\
0, & \text{otherwise}
\end{cases} \quad (2.8)$$

contains only two subsequences, namely, $[y_{n,k}] = 0$ for each $n$ and $k$, and

$$z_{n,k} := \begin{cases} 
1, & \text{if } n = k = 0, \\
0, & \text{otherwise};
\end{cases} \quad (2.9)$$

neither subsequences is $x$.

The following propositions are easily verified.

**Proposition 2.22.** If every subsequence of $x = [x_{k,l}]$ is $P$-convergent, then $x$ is $P$-convergent.

**Proposition 2.23.** The double sequence $x$ is $P$-convergent to $L$ if and only if every subsequence of $x$ is $P$-convergent to $L$.

**Definition 2.24.** The double sequence $y$ contains an $\epsilon$-Pringsheim-copy of $x$ provided that $y$ contains a subsequence $y_{n_i,k_j}$ such that $|y_{n_i,k_j} - x_{i,j}| < \epsilon_{i,j}$, for $i,j = 1,2,\ldots$.

**Example 2.25.** Let

$$x_{n,k} := \begin{cases} 
(-1)^n, & \text{if } k = n, \\
0, & \text{otherwise},
\end{cases} \quad (2.10)$$

and let $P$-$\lim_{n,k} \epsilon_{n,k} = 0$ with

$$y_{n,k} := \begin{cases} 
(-1)^n, & \text{if } k = n, \\
\epsilon_{n,k}, & \text{otherwise}.
\end{cases} \quad (2.11)$$

Observe that, not only does $y$ contain an $\epsilon$-Pringsheim-copy of $x$, but $y$ itself is an $\epsilon$-Pringsheim-copy of $x$. 
**Definition 2.26.** The double sequence \( y \) is a *stretching* of \( x \) provided that there exist two increasing index sequences \( \{R_i\}_{i=0}^\infty \) and \( \{S_j\}_{j=0}^\infty \) of integers such that

\[
y_{n,k} := \begin{cases} 
R_0 = S_0 = 1, \\
x_{n,i}, & \text{if } R_{i-1} \leq k < R_i, \\
x_{j,k}, & \text{if } S_{j-1} \leq n < S_j, \\
i, j = 1, 2, \ldots
\end{cases} \tag{2.12}
\]

**Remark 2.27.** This definition demonstrates the procedure which is used to construct a stretching of a double sequence \( x \). This procedure uses a sequence of stages to construct the stretching of \( x \). These stages are constructed using a sequence of abutting rows and columns of \( x \). These rows and columns are constructed as follows.

**Stage 1.** Begin by repeating the first row of \( x \) \( R_1 \) times and denote the resulting double sequence by \( y^{1,0} \) then repeat the first column of \( y^{1,0} \) \( S_1 \) times resulting in \( y^{1,1} \).

**Stage 2.** Begin by repeating the \( R_1 + 1 \) row of \( y^{1,1} \), \( R_2 - R_1 \) times which yields \( y^{2,1} \) then repeat the \( S_1 + 1 \) column of \( y^{2,1} \), \( S_2 - S_1 \) times which yields \( y^{2,2} \).

\[
\vdots
\]

**Stage \( i \).** Begin by repeating the \( 1 + \sum_{p=1}^{i-1} R_p \) row of \( y^{i-1,i-1} \), \( R_i - R_{i-1} \) times which yields \( y^{i,i-1} \) then repeat the \( 1 + \sum_{q=1}^{i-1} S_q \) column of \( y^{i,i-1} \), \( S_i - S_{i-1} \) times which yields \( y^{i,i} \). Note that in each stage we repeat the number of rows and then repeat the number of columns. However the resulting stretching \( y \) of \( x \) is the same, if we first repeat the number of columns and then repeat the numbers of rows. Also note that every sequence itself is a stretching of itself and the sequences that induce this kind of stretching are \( R_i = i \) and \( S_j = j \).

**Example 2.28.** The sequence

\[
\begin{array}{ccccccccccc}
x_{1,1} & x_{1,1} & x_{1,1} & x_{1,2} & x_{1,2} & x_{1,3} & x_{1,3} & \cdots \\
x_{1,1} & x_{1,1} & x_{1,1} & x_{1,2} & x_{1,2} & x_{1,3} & x_{1,3} & \cdots \\
x_{1,1} & x_{1,1} & x_{1,1} & x_{1,2} & x_{1,2} & x_{1,3} & x_{1,3} & \cdots \\
x_{2,1} & x_{2,1} & x_{2,1} & x_{2,2} & x_{2,2} & x_{2,3} & x_{2,3} & \cdots \\
x_{2,1} & x_{2,1} & x_{2,1} & x_{2,2} & x_{2,2} & x_{2,3} & x_{2,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

is a stretching of \( x \) induced by \( R_i = 3i \) and \( S_j = 3j \).

3. **Main results.** The following theorem is given its name because of its similarity to the copy theorem of Dawson in [1].

**Theorem 3.1** (extended copy theorem). If each of \( A \) and \( T \) is an RH-regular matrix, and \( x \) is any bounded double complex sequence with \( \epsilon \) being any bounded positive term
double sequence with $P\lim_{i,j} c_{i,j} = 0$, then there exists a stretching $\gamma$ of $x$ such that $T(Ay)$ exists and contains an $e$-Pringsheim-copy of $x$.

**Proof.** We begin by introducing a few notations which are used only in this proof. Let

$$
||A|| := \sup_{m,n>\bar{B}} \left( \sum_{k,l} |a_{m,n,k,l}| \right) < K_A, \quad ||T|| := \sup_{m,n>\bar{B}} \left( \sum_{k,l} |t_{m,n,k,l}| \right) < K_T,
$$

$$
M_{i,j} := 1 + \sum_{k,l=1}^{i,j} |x_{k,l}|, \quad \delta_{i,j} := \min_{i,j} \left\{ \frac{\epsilon_{i,j}}{1} \leq k \leq i \cup 1 \leq l \leq j \right\},
$$

$$
K := K_A + K_T + \max_{i,j} \left\{ \frac{\epsilon_{i,j}}{1} \leq k \leq i \cup 1 \leq l \leq j \right\} + 1, \quad Q_{i,j} := K_{M_{i,j}} + 1,
$$

$$
c_{i,j}(r,s) := \left\{ \frac{(k,l)}{1} \leq k < r_i \cup 1 \leq l < s_j \right\},
$$

$$
\tilde{c}_{i,j}(r,s) := \left\{ \frac{(k,l)}{r_i} \leq k < \infty \cup s_j \leq l < \infty \right\}, \quad \tilde{b}_{i,j}(r,s) := c_{i,j}(r,s) \setminus c_{i-1,j-1}(r,s).
$$

Then by (RH2) there exist $m_{\alpha_1}$ and $n_{\beta_1}$ such that for $m > m_{\alpha_1} > \bar{B}$ and $n > n_{\beta_1} > \bar{B}$, where $\bar{B}$ is defined by the sixth RH-condition,

$$
\left| \sum_{k,l=1}^{\infty,\infty} a_{m,n,k,l} - 1 \right| < \frac{\delta_{\alpha_1,\beta_1}}{16Q_{\alpha_1,\beta_1}}.
$$

(3.2)

Also by (RH1) and (RH2) there exist $a_{\alpha_1}$ and $b_{\beta_1}$ such that

$$
\sum_{(k,l) \in c_{\alpha_1,\beta_1}(m,n)} |t_{a_{\alpha_1},b_{\beta_1},k,l}| < \frac{\delta_{\alpha_1,\beta_1}}{8Q_{\alpha_1,\beta_1}}, \quad \left| \sum_{k,l=1}^{\infty,\infty} t_{a_{\alpha_1},b_{\beta_1},k,l} - 1 \right| < \frac{\delta_{\alpha_1,\beta_1}}{8Q_{\alpha_1,\beta_1}}.
$$

(3.3)

In addition, there exist $\tilde{m}_{\alpha_1}, \tilde{n}_{\beta_1}, \alpha_2,$ and $\beta_2$ such that if $1 \leq \psi \leq a_{\alpha_1}$ and $1 \leq \omega \leq b_{\beta_1}$, then

$$
\sum_{(k,l) \in c_{\alpha_1,\beta_1}(m,n)} |t_{\psi,\omega,k,l}| < \frac{\delta_{\alpha_1,\beta_1}}{16Q_{\alpha_2,\beta_2}}.
$$

(3.4)

Also, there exist $r_{\alpha_1} > 1$ and $s_{\beta_1} > 1$ such that if $1 \leq m \leq \tilde{m}_{\alpha_1}$ and $1 \leq n \leq \tilde{n}_{\beta_1}$ then

$$
\sum_{(k,l) \in \tilde{c}_{\alpha_1,\beta_1}(r,s)} |a_{m,n,k,l}| < \frac{\delta_{\alpha_1,\beta_1}}{16Q_{\alpha_2,\beta_2}}.
$$

(3.5)

Now, without loss of generality, we set $\alpha_p = p$ and $\beta_q = q$. Having chosen

$$
\left\{ \begin{array}{ll}
&m_p, \tilde{m}_p, a_p, r_p \\
n_q, \tilde{n}_q, b_q, s_q \\
p = 0, q = 0
\end{array} \right. \left( \begin{array}{c}
i-1, j-1 \\
p=0, q=0
\end{array} \right)
$$

(3.6)

with $m_0 = n_0 = \tilde{m}_0 = \tilde{n}_0 = a_0 = b_0 = r_0 = s_0 = 1$, now choose $m_i > \tilde{m}_{i-1}$ and $n_j > \tilde{n}_{j-1}$ such that if $m > m_i$ and $n > n_j$ then

$$
\left| \sum_{(k,l) \in \tilde{c}_{i-1,j-1}(r,s)} a_{m,n,k,l} - 1 \right| < \frac{\delta_{i,j}}{16Q_{i,j}2^{i+j}}.
$$

(3.7)
Also choose \( a_i > a_{i-1} \) and \( b_j > b_{j-1} \) such that

\[
\sum_{(k,l) \in \overline{c}_{i,j}(m,n)} |t_{a_i,b_j,k,l}| < \frac{\delta_{i,j}}{2^{i+j+1}Q_{i,j}}, \quad \sum_{(k,l) \in \overline{c}_{i,j}(m,n)} t_{a_i,b_j,k,l} - 1 < \frac{\delta_{i,j}}{2^{2+i+j}Q_{i,j}}.
\] (3.9)

Next choose \( \tilde{m}_i > m_i \) and \( \tilde{n}_j > n_j \) such that if \( 1 \leq p \leq a_i \) and \( 1 \leq q \leq b_j \) then

\[
\sum_{(k,l) \in \overline{c}_{i,j}(\tilde{m},\tilde{n})} |t_{p,q,k,l}| < \frac{\delta_{i,j}}{2^{p+q+i+j+1}Q_{i,j}},
\] (3.10)

where \( m_i, n_j, \tilde{m}_i, \tilde{n}_j \) are chosen using \((RH_1), (RH_2), (RH_3), \) and \((RH_4)\) such that if \( 1 \leq p \leq j-1 \) and \( 1 \leq q \leq i-1 \) the following is obtained:

\[
\left| \sum_{(k,l) \in \overline{c}_{i,j}(m,n)} a_{m,n,k,l} \right| \leq \frac{\delta_{p,j}}{8Q_{p,j}2^{p+j}}, \quad \left| \sum_{(k,l) \in \overline{b}_{i,j}(r,s)} a_{m,n,k,l} \right| \leq \frac{\delta_{i,q}}{8Q_{i,q}2^{i+j}}.
\] (3.12)

Therefore by (3.9) and (3.10) we have

\[
\left| \sum_{(k,l) \in \overline{c}_{i,j}(\tilde{m},\tilde{n}) \setminus \overline{c}_{i,j}(m,n)} t_{a_i,b_j,k,l} - 1 \right| \leq \frac{\delta_{i,j}}{4Q_{i,j}},
\] (3.13)

and by (3.7), (3.8), and (3.11) we also obtain

\[
\left| \sum_{(k,l) \in \overline{b}_{i,j}(r,s)} a_{m,n,k,l} - 1 \right| < \frac{\delta_{i,j}}{8Q_{i,j}2^{i+j}},
\] (3.14)

where \( m_i \leq m \leq \tilde{m}_i \) and \( n_j \leq n \leq \tilde{n}_j \). Let \( \{y_{k,l}\} \) be the stretching of \( x \) induced by \( \{r_i\} \) and \( \{s_j\} \). Since

\[
(Ay)_{m,n} - x_{i,j} = \sum_{k,l=1}^{r_{i-1},s_{j-1}-1} a_{m,n,k,l}y_{k,l} + \sum_{(k,l) \in \overline{b}_{i,j}(r,s)} a_{m,n,k,l}y_{k,l} - x_{i,j}
\] (3.15)

if \( i,j > 1 \), with \( m_i \leq m \leq \tilde{m}_i \) and \( n_j \leq n \leq \tilde{n}_j \) the following is obtained:

\[
\left| \sum_{k,l=1}^{r_{i-1},s_{j-1}-1} a_{m,n,k,l}y_{k,l} \right| \leq \max \left\{ \frac{|x_{k,l}|}{1} \leq k \leq i-1 \cup l \leq j-1 \right\} \sum_{k,l=1}^{r_{i-1},s_{j-1}-1} |a_{m,n,k,l}y_{k,l}|.
\] (3.16)
By (3.8),
\[
\left| \sum_{k,l=1}^{r_{i-1,j-1}} a_{m,n,k,l} y_{k,l} \right| \leq \max \left\{ \frac{|x_{k,l}|}{1} \leq k \leq i-1 \cup 1 \leq l \leq j-1 \right\} \frac{\delta_{i,j}}{8Q_{i-1,j-1}}. \tag{3.17}
\]
Since
\[
Q_{i-1,j-1} = K \left( 1 + \sum_{k,l=1}^{i-1-j-1} |x_{k,l}| \right) + 1 \geq K \max \left\{ \frac{|x_{k,l}|}{1} \leq k \leq i-1 \cup 1 \leq l \leq j-1 \right\}, \tag{3.18}
\]
the following holds:
\[
\left| \sum_{k,l=1}^{r_{i-1,j-1}} a_{m,n,k,l} y_{k,l} \right| \leq \frac{\delta_{i,j}}{8K}, \tag{3.19}
\]
the following also is obtained:
\[
\left| \sum_{p,q=i+1,j+1}^{\infty} a_{m,n,k,l} y_{k,l} \right| \leq \sum_{p,q=i+1,j+1}^{\infty} |x_{p,q}| \sum_{(k,l) \in \bar{b}_{p,q}(r,s)} |a_{m,n,k,l}| \leq \frac{\delta_{i,j}}{24K} \sum_{p,q=i+1,j+1}^{\infty} \frac{1}{2^{p+q}} \leq \frac{\delta_{i,j}}{8K}, \tag{3.20}
\]
because
\[
\sum_{k,l=r_{p,q},s}^{\infty} |a_{m,n,k,l}| \leq \frac{\delta_{p-1,q-1}}{24+p+q} Q_{p,q} |x_{p,q}| < \frac{1}{K}. \tag{3.21}
\]
Therefore by (3.11),
\[
\left| \sum_{(k,l) \in \bar{b}_{i,j}(r,s)} a_{m,n,k,l} y_{k,l} - x_{i,j} \right| \leq \sum_{q=1}^{i-1} |x_{i,q}| \sum_{(k,l) \in \bar{b}_{i,q}(r,s)} |a_{m,n,k,l}| \tag{3.22}
\]
\[
+ \sum_{p=1}^{j-1} |x_{p,j}| \sum_{(k,l) \in \bar{b}_{p,j}(r,s)} |a_{m,n,k,l}| + |x_{i,j}| \sum_{(k,l) \in \bar{b}_{i,j}(r,s)} a_{m,n,k,l}^{-1} \leq \sum_{p,q=1,1}^{i,j} \frac{|x_{i,j}|}{Q_{i,j}} \frac{\delta_{p,q}}{2^{p+q+3}} \leq \frac{\delta_{i,j}}{K8} \sum_{p,q=1,1}^{i,j} \frac{1}{2^{p+q}} = \frac{\delta_{i,j}}{2K^2}.
\]
Therefore,
\[
|(Ay)_{m,n} - x_{i,j}| \leq \frac{\delta_{i,j}}{K8} + \frac{\delta_{i,j}}{K4} + \frac{\delta_{i,j}}{K2} < \frac{\delta_{i,j}}{2K}. \tag{3.23}
\]
Note that the inequality (3.23) is true for \( m_1 \leq m \leq \tilde{m}_1 \) and \( n_1 \leq n \leq \tilde{n}_1 \), and also this inequality is true for \( i, j \geq 1 \) with \( m_i \leq m \leq \tilde{m}_i \) and \( n_j \leq n \leq \tilde{n}_j \). Hence

\[
(Ay)_{m,n} = x_{i,j} + u_{i,j},
\]

where \( |u_{i,j}| \leq \delta_{i,j}/2K \). Note that if \( \tilde{m}_{i-1} \leq m \leq m_i \) and \( \tilde{n}_{j-1} \leq n \leq n_j \), then the following is obtained:

\[
\left| (Ay)_{m,n} \right| \leq \left| \sum_{k,l=1}^{r_1-1,s_1-1} a_{m,n,k,l} y_{k,l} \right| + \left| \sum_{p,q=i+1,j+1}^{\infty} \sum_{k,l \in b_{p,q}(r,s)} a_{m,n,k,l} y_{k,l} \right|
\]

\[
\leq \max \left\{ \frac{|x_{i,j}|}{1} \leq k \leq i \leq j \right\} \sum_{k,l=1}^{r_1-1,s_1-1} |a_{m,n,k,l}|
\]

\[
+ \sum_{p,q=i+1,j+1}^{\infty} |x_{i,j}| \sum_{k,l \in b_{p,q}(r,s)} |a_{m,n,k,l}|
\]

\[
\leq Km_{i,j} + \sum_{p,q=i+1,j+1}^{\infty} |x_{i,j}| \frac{\delta_{p,q}}{2^{p+q}Q_{p+1,q+1}}
\]

\[
\leq Km_{i,j} + \frac{\delta_{i,j}}{K} + Km_{i,j} \leq Km_{i,j} + 1 = Q_{i,j}.
\]

Also, if \( m_{i-1} \leq m \leq m_i \) and \( n_{j-1} \leq n \leq n_j \) then

\[
\left| \sum_{k,l=1}^{\infty} a_{m,n,k,l} y_{k,l} \right| \leq \left| (Ay)_{m,n} - x_{i,j} \right| + |x_{i,j}|
\]

\[
\leq \frac{\delta_{i,j}}{2K} + Km_{i,j} \leq Km_{i,j} + 1 = Q_{i,j}.
\]

By using (3.25) we now show the existence of \( T(Ay) \). If \( a_{i-1} < m \leq a_i \) and \( b_{j-1} < n \leq b_j \) then

\[
\left| \sum_{k,l=\tilde{m}_i+1,\tilde{n}_j+1}^{\infty} t_{m,n,k,l} (Ay)_{k,l} \right| \leq \sum_{r,s=i,j}^{\infty} \sum_{(p,q) \in b_{r+1,s+1}^1 (m,n)} \left| t_{m,n,p,q} (Ay)_{p,q} \right|
\]

\[
\leq \sum_{r,s=i,j}^{\infty} Q_{r+s+1} \sum_{(p,q) \in b_{r+1,s+1}^1 (m,n)} \left| t_{m,n,p,q} \right|
\]

\[
\leq \sum_{r,s=i,j}^{\infty} Q_{r+s+1} \frac{\delta_{r,s}}{2^{2r+s}Q_{r+1,s+1}}
\]

\[
\leq \delta_{i,j} \frac{1}{4} \sum_{r,s=1}^{\infty} \frac{1}{2^{r+s}} < \frac{\delta_{i,j}}{4}.
\]
Therefore $T(Ay)$ exists. Also, by (3.25) we now show that $T(Ay)$ contains an $\epsilon$-Pringsheim-copy of $x$. First note that

\[
\begin{align*}
\sum_{k,l=1}^{\infty,\infty} t_{a_{i,j},k,l}(Ay)_{k,l} - x_{i,j} & \leq \sum_{k,l=1}^{m_i-1,n_j-1} |t_{a_{i,j},k,l}(Ay)_{k,l}| \\
& + \sum_{(k,l) \in \bar{b}_{i,j}(r,s)} |t_{a_{i,j},k,l}(Ay)_{k,l} - x_{i,j}| \\
& + \sum_{k,l=m_i+1,m_j+1}^{\infty,\infty} |t_{m,n,k,l}(Ay)_{k,l}|,
\end{align*}
\]

with

\[
\begin{align*}
\sum_{k,l=1}^{m_i-1,n_j-1} |t_{a_{i,j},k,l}(Ay)_{k,l}| & = \sum_{k,l=1}^{m_i-1,n_j-1} |t_{a_{i,j},k,l}| Q_{i,j} \leq Q_{i,j} \frac{\delta_{i,j}}{8Q_{i,j}} = \frac{\delta_{i,j}}{8}, \\
\sum_{(k,l) \in \bar{b}_{i,j}(r,s)} |t_{a_{i,j},k,l}(Ay)_{k,l} - x_{i,j}| & \leq |x_{i,j}| \sum_{(k,l) \in \bar{b}_{i,j}(r,s)} |t_{a_{i,j},k,l} - 1| + \sum_{(k,l) \in \bar{b}_{i,j}(r,s)} |t_{a_{i,j},k,l}| |u_{i,j}| \\
& \leq \frac{|x_{i,j}| \delta_{i,j}}{4Q_{i,j}^2} + \frac{\delta_{i,j}}{4K} \sum_{(k,l) \in \bar{b}_{i,j}(r,s)} |t_{a_{i,j},k,l}| \\
& \leq \frac{\delta_{i,j}}{2}, \\
\sum_{k,l=m_i+1,m_j+1}^{\infty,\infty} |t_{m,n,k,l}(Ay)_{k,l}| & \leq \sum_{r,s=i,j}^{\infty,\infty} \sum_{(p,q) \in \bar{b}_{r,s+1}(m,n)} |t_{a_{i,j},p,q}(Ay)_{p,q}| \\
& \leq \sum_{r,s=i,j}^{\infty,\infty} Q_{r+1,s+1} \sum_{(p,q) \in \bar{b}_{r,s+1}(m,n)} |t_{a_{i,j},p,q}| \\
& \leq \sum_{r,s=i,j}^{\infty,\infty} Q_{r+1,s+1} \frac{\delta_{r,s}}{2^{2r+s}} \frac{1}{Q_{r+1,s+1}} \leq \frac{\delta_{i,j}}{4}.
\end{align*}
\]

Hence,

\[
\sum_{k,l=1}^{\infty,\infty} t_{m,n,k,l}(Ay)_{k,l} - x_{i,j} \leq \frac{\delta_{i,j}}{4} + \frac{\delta_{i,j}}{2} + \frac{\delta_{i,j}}{8} < \delta_{i,j} \leq \epsilon_{i,j}.
\]

This completes the proof of the extended copy theorem. □
The next two results are immediate corollaries of the extended copy theorem.

**Corollary 3.2.** If $T$ is any RH-regular matrix summability method and $A$ is an RH-regular matrix such that $Ay$ is $T$-summable for every stretching $y$ of $x$, then $x$ is $P$-convergent.

**Corollary 3.3.** If $T$ is any RH-regular matrix summability method and $A$ is an RH-regular matrix such that $Ay$ is absolutely $T$-summable for every stretching $y$ of $x$, then $x$ is $P$-convergent.

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**References**


Richard F. Patterson: Department of Mathematics and Statistics, University of North Florida, Building 11, Jacksonville, FL 32224, USA

E-mail address: rpatters@unf.edu