ON Q-ALGEBRAS

JOSEPH NEGGERS, SUN SHIN AHN, and HEE SIK KIM

(Received 29 January 2001)

Abstract. We introduce a new notion, called a Q-algebra, which is a generalization of the idea of BCH/BCI/BCK-algebras and we generalize some theorems discussed in BCI-algebras. Moreover, we introduce the notion of "quadratic" Q-algebra, and show that every quadratic Q-algebra \((X; *, e), e \in X\), has a product of the form \(x \ast y = x - y + e\), where \(x, y \in X\) when \(X\) is a field with \(|X| \geq 3\).

2000 Mathematics Subject Classification. 06F35, 03G25.

1. Introduction. Imai and Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras (see [4, 5]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [2, 3] Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Neggers and Kim (see [8]) introduced the notion of \(d\)-algebras, that is, (I) \(x \ast x = e\); (IX) \(e \ast x = e\); (VI) \(x \ast y = e\) and \(y \ast x = e\) imply \(x = y\), which is another useful generalization of BCH/BCI/BCK-algebras, after which they investigated several relations between \(d\)-algebras and BCK-algebras, as well as other relations between \(d\)-algebras and oriented digraphs. At the same time, Jun, Roh, and Kim [6] introduced a new notion, called a BH-algebra, that is, (I) \(x \ast x = e\); (II) \(x \ast e = x\); (VI) \(x \ast y = e\) and \(y \ast x = e\) imply \(x = y\), which is a generalization of \(BCH/BCI/BCK\)-algebras, and they showed that there is a maximal ideal in bounded BH-algebras. We introduce a new notion, called a Q-algebra, which is a generalization of BCH/BCI/BCK-algebras and generalize some theorems from the theory of BCI-algebras. Moreover, we introduce the notion of “quadratic” Q-algebra, and obtain the result that every quadratic Q-algebra \((X; *, e), e \in X\), is of the form \(x \ast y = x - y + e\), where \(x, y \in X\) and \(X\) is a field with \(|X| \geq 3\), that is, the product is linear in a special way.

2. Q-algebras. A Q-algebra is a nonempty set \(X\) with a constant 0 and a binary operation “\(*\)” satisfying axioms:

(I) \(x \ast x = 0\),

(II) \(x \ast 0 = x\),

(III) \((x \ast y) \ast z = (x \ast z) \ast y\) for all \(x, y, z \in X\).

For brevity we also call \(X\) a Q-algebra. In \(X\) we can define a binary relation \(\leq\) by \(x \leq y\) if and only if \(x \ast y = 0\). Recently, Ahn and Kim [1] introduced the notion of QS-algebras. A Q-algebra \(X\) is said to be a QS-algebra if it satisfies the additional relation:

(IV) \((x \ast y) \ast (x \ast z) = z \ast y\), for any \(x, y, z \in X\).
**Example 2.1.** Let $\mathbb{Z}$ be the set of all integers and let $n\mathbb{Z} := \{ nz \mid z \in \mathbb{Z} \}$ where $n \in \mathbb{Z}$. Then $(\mathbb{Z}; -, 0)$ and $(n\mathbb{Z}; -, 0)$ are $Q$-algebras, where “-” is the usual subtraction of integers.

**Example 2.2.** Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(X; *, 0)$ is a $Q$-algebra, which is not a $BCH/BCI/BCK$-algebra.

Neggers and Kim [7] introduced the related notion of $B$-algebra, that is, algebras $(X; *, 0)$ which satisfy (I) $x * x = 0$; (II) $x * 0 = x$; (V) $(x * y) * z = x * (z * (0 * y))$, for any $x, y, z \in X$. It is easy to see that $B$-algebras and $Q$-algebras are different notions. For example, Example 2.2 is a $Q$-algebra, but not a $B$-algebra, since $(3 * 2) * 1 = 0 \neq 3 = 3 * (1 * (0 * 2))$. Consider the following example. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(X; *, 0)$ is a $B$-algebra (see [7]), but not a $Q$-algebra, since $(5 * 3) * 4 = 3 \neq 4 = (5 * 4) * 3$.

**Proposition 2.3.** If $(X; *, 0)$ is a $Q$-algebra, then (VII) $(x * (x * y)) * y = 0$, for any $x, y \in X$.

**Proof.** By (I) and (III), $(x * (x * y)) * y = (x * y) * (x * y) = 0$. □

We now investigate some relations between $Q$-algebras and $BCH$-algebras (also $BCK/BCI$-algebras). The following theorems are easily proven, and we omit their proofs.

**Theorem 2.4.** Every $BCH$-algebra $X$ is a $Q$-algebra. Every $Q$-algebra $X$ satisfying condition (VI) is a BCH-algebra.

**Theorem 2.5.** Every $Q$-algebra satisfying condition (IV) and (VI) is a BCI-algebra.
Theorem 2.6. Every $Q$-algebra $X$ satisfying conditions (V), (VI), and (VIII) $(x \ast y) \ast x = 0$ for any $x, y \in X$, is a BCK-algebra.

Theorem 2.7. Every $Q$-algebra $X$ satisfying $x \ast (x \ast y) = x \ast y$ for all $x, y, z \in X$, is a trivial algebra.

Proof. Putting $x = y$ in the equation $x \ast (x \ast y) = x \ast y$, we obtain $x \ast 0 = 0$. By (II) $x = 0$. Hence $X$ is a trivial algebra.

The following example shows that a $Q$-algebra may not satisfy the associative law.

Example 2.8. (a) Let $X := \{0, 1, 2\}$ with the table as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $X$ is a $Q$-algebra, but associativity does not hold, since $(0 \ast 1) \ast 2 = 0 \neq 1 = 0 \ast (1 \ast 2)$.

(b) Let $\mathbb{Z}$ and $\mathbb{R}$ be the set of all integers and real numbers, respectively. Then $(\mathbb{Z}; -, 0)$ and $(\mathbb{R}; \div, 1)$ are nonassociative $Q$-algebras where “-” is the usual subtraction and “\div” is the usual division.

Theorem 2.9. Every $Q$-algebra $(X; \ast, 0)$ satisfying the associative law is a group under the operation “\ast”.

Proof. Putting $x = y = z$ in the equation $(x \ast y) \ast z = x \ast (y \ast z)$ and using (I) and (II), we obtain $0 \ast x = x \ast 0 = x$. This means that 0 is the zero element of $X$. By (I), every element $x$ of $X$ has as its inverse the element $x$ itself. Therefore $(X; \ast)$ is a group.

3. The $G$-part of $Q$-algebras. In this section, we investigate the properties of the $G$-part in $Q$-algebras.

Lemma 3.1. If $(X; \ast, 0)$ is a $Q$-algebra and $a \ast b = a \ast c, a, b, c \in X$, then $0 \ast b = 0 \ast c$.

Proof. By (I) and (II) $(a \ast b) \ast a = (a \ast a) \ast b = 0 \ast b$ and $(a \ast c) \ast a = (a \ast a) \ast c = 0 \ast c$. Since $a \ast b = a \ast c$, $0 \ast b = 0 \ast c$.

Definition 3.2. Let $(X; \ast, 0)$ be a $Q$-algebra. For any nonempty subset $S$ of $X$, we define

$$G(S) := \{x \in S \mid 0 \ast x = x\}. \quad (3.1)$$

In particular, if $S = X$ then we say that $G(X)$ is the $G$-part of $X$.

Corollary 3.3. A left cancellation law holds in $G(X)$.

Proof. Let $a, b, c \in G(X)$ with $a \ast b = a \ast c$. By Lemma 3.1, $0 \ast b = 0 \ast c$. Since $b, c \in G(X)$, we obtain $b = c$. □
**Proposition 3.4.** Let \((X;\ast,0)\) be a \(Q\)-algebra. Then \(x \in G(X)\) if and only if \(0 \ast x \in G(X)\).

**Proof.** If \(x \in G(X)\), then \(0 \ast x = x\) and \(0 \ast (0 \ast x) = 0 \ast x\). Hence \(0 \ast x \in G(X)\).

Conversely, if \(0 \ast x \in G(X)\), then \(0 \ast (0 \ast x) = 0 \ast x\). By applying Corollary 3.3, we obtain \(0 \ast x = x\). Therefore \(x \in G(X)\).

For any \(Q\)-algebra \((X;\ast,0)\), the set

\[
B(X) := \{ x \in X \mid 0 \ast x = 0 \}
\]

is called the **\(p\)-radical** of \(X\). If \(B(X) = \{0\}\), then we say that \(X\) is a **\(p\)-semisimple** \(Q\)-algebra. The following property is obvious.

\[ (\text{IX}) \ G(X) \cap B(X) = \{0\}. \]

**Proposition 3.5.** If \((X;\ast,0)\) is a \(Q\)-algebra and \(x,y \in X\), then

\[ y \in B(X) \iff (x \ast y) \ast x = 0. \]

**Proof.** By (I) and (III) \((x \ast y) \ast x = (x \ast x) \ast y = 0 \ast y = 0\) if and only if \(y \in B(X)\).

**Definition 3.6.** Let \((X;\ast,0)\) be a \(Q\)-algebra and \(I(\neq \emptyset) \subseteq X\). The set \(I\) is called an **ideal** of \(X\) if for any \(x,y,z \in X\),

1. \(0 \in I\),
2. \(x \ast y \in I\) and \(y \in I\) imply \(x \in I\).

Obviously, \(\{0\}\) and \(X\) are ideals of \(X\). We call \(\{0\}\) and \(X\) the **zero ideal** and the **trivial ideal** of \(X\), respectively. An ideal \(I\) is said to be **proper** if \(I \neq X\).

In Example 2.2 the set \(I := \{0,1,2\}\) is an ideal of \(X\).

**Proposition 3.7.** Let \((X;\ast,0)\) be a \(Q\)-algebra. Then \(B(X)\) is an ideal of \(X\).

**Proof.** Since \((0 \ast 0) \ast 0 = 0\), by Proposition 3.5, \(0 \in B(X)\). Let \(x \ast y \in B(X)\) and \(y \in B(X)\). Then by Proposition 3.5, \((x \ast y) \ast x = (x \ast x) \ast y = 0 \ast y = 0\). By (III), \((x \ast y) \ast (x \ast y)) \ast x = 0 \ast x = 0\). Hence \(x \in B(X)\). Therefore \(B(X)\) is an ideal of \(X\).

**Proposition 3.8.** If \(S\) is a subalgebra of a \(Q\)-algebra \((X;\ast,0)\), then \(G(X) \cap S = G(S)\).

**Proof.** It is obvious that \(G(X) \cap S \subseteq G(S)\). If \(x \in G(S)\), then \(0 \ast x = x\) and \(x \in S \subseteq X\). Then \(x \in G(X)\) and so \(x \in G(X) \cap S\), which proves the proposition.

**Theorem 3.9.** Let \((X;\ast,0)\) be a \(Q\)-algebra. If \(G(X) = X\), then \(X\) is \(p\)-semisimple.

**Proof.** Assume that \(G(X) = X\). By (X), \(\{0\} = G(X) \cap B(X) = X \cap B(X) = B(X)\). Hence \(X\) is \(p\)-semisimple.

**Theorem 3.10.** If \((X;\ast,0)\) is a \(Q\)-algebra of order \(3\), then \(|G(X)| \neq 3\), that is, \(G(X) \neq X\).

**Proof.** For the sake of convenience, let \(X = \{0,a,b\}\) be a \(Q\)-algebra. Assume that \(|G(X)| = 3\), that is, \(G(X) = X\). Then \(0 \ast 0 = 0\), \(0 \ast a = a\), and \(0 \ast b = b\). From \(x \ast x = 0\) and \(x \ast 0 = x\), it follows that \(a \ast a = 0\), \(b \ast b = 0\), \(a \ast 0 = a\), and \(b \ast 0 = b\). Now let \(a \ast b = 0\). Then \(0\), \(a\), and \(b\) are candidates of the computation. If \(b \ast a = 0\), then
implicative, it follows that $a \ast b = 0 = b \ast a$ and so $(a \ast b) \ast a = (b \ast a) \ast a$. By (III), $(a \ast a) \ast b = (b \ast a) \ast a$. Hence $0 \ast 0 = 0 \ast a$. By the cancellation law in $G(X)$, $b = a$, a contradiction. If $b \ast a = a$, then $a = b \ast a = (0 \ast b) \ast a = (0 \ast a) \ast b = a \ast b = 0$, a contradiction. For the case $b \ast a = b$, we have $b = b \ast a = (0 \ast b) \ast a = (0 \ast a) \ast b = a \ast b = 0$, which is also a contradiction. Next, if $a \ast b = a$, then $(a \ast (a \ast b)) \ast b = (a \ast a) \ast b = 0 \ast b = b \neq 0$. This leads to the conclusion that Proposition 2.3 does not hold, a contradiction. Finally, let $a \ast b = b$. If $b \ast a = 0$, then $b = a \ast b = (0 \ast a) \ast b = (0 \ast b) \ast a = b \ast a = 0$, a contradiction. If $b \ast a = a$, $b = a \ast b = (0 \ast a) \ast b = (0 \ast b) \ast a = b \ast a = 0$, a contradiction. For the case $b \ast a = b$, we have $a = 0 \ast a = (b \ast b) \ast a = (b \ast a) \ast b = b \ast b = 0$, which is again a contradiction. This completes the proof.

**Proposition 3.11.** If $(X; \ast, 0)$ is a $Q$-algebra of order 2, then in every case the $G$-part $G(X)$ of $X$ is an ideal of $X$.

**Proof.** Let $|X| = 2$. Then either $G(X) = \{0\}$ or $G(X) = X$. In either case, $G(X)$ is an ideal of $X$.

**Theorem 3.12.** Let $(X; \ast, 0)$ be a $Q$-algebra of order 3. Then $G(X)$ is an ideal of $X$ if and only if $|G(X)| = 1$.

**Proof.** Let $X := \{0, a, b\}$ be a $Q$-algebra. If $|G(X)| = 1$, then $G(X) = \{0\}$ is the trivial ideal of $X$.

Conversely, assume that $G(X)$ is an ideal of $X$. By Theorem 3.10, we know that either $|G(X)| = 1$ or $|G(X)| = 2$. Suppose that $|G(X)| = 2$. Then either $G(X) = \{0, a\}$ or $G(X) = \{0, b\}$. If $G(X) = \{0, a\}$, then $b \ast a \notin G(X)$ because $G(X)$ is an ideal of $X$. Hence $b \ast a = b$. Then $a = 0 \ast a = (b \ast b) \ast a = (b \ast a) \ast b = b \ast b = 0$, which is a contradiction. Similarly, $G(X) = \{0, b\}$ leads to a contradiction. Therefore $|G(X)| = 2$ and so $|G(X)| = 1$.

**Definition 3.13.** An ideal $I$ of a $Q$-algebra $(X; \ast, 0)$ is said to be implicative if $(x \ast y) \ast z \in I$ and $y \ast z \in I$, then $x \ast z \in I$, for any $x, y, z \in X$.

**Theorem 3.14.** Let $(X; \ast, 0)$ be a $Q$-algebra and let $I$ be an implicative ideal of $X$. Then $I$ contains the $G$-part $G(X)$ of $X$.

**Proof.** If $x \in G(X)$, then $(0 \ast x) \ast x = x \ast x = 0 \in I$ and $x \ast x = 0 \in I$. Since $I$ is implicative, it follows that $x = 0 \ast x \in I$. Hence $G(X) \subseteq I$.

**Definition 3.15.** Let $X$ and $Y$ be $Q$-algebras. A mapping $f : X \to Y$ is called a homomorphism if

$$f(x \ast y) = f(x) \ast f(y), \quad \forall x, y \in X.$$  \hfill (3.4)

A homomorphism $f$ is called a monomorphism (resp., epimorphism) if it is injective (resp., surjective). A bijective homomorphism is called an isomorphism. Two $Q$-algebras $X$ and $Y$ are said to be isomorphic, written by $X \cong Y$, if there exists an isomorphism $f : X \to Y$. For any homomorphism $f : X \to Y$, the set $\{x \in X \mid f(x) = 0\}$ is called the kernel of $f$, denoted by $\text{Ker}(f)$ and the set $\{f(x) \mid x \in X\}$ is called the image of $f$, denoted by $\text{Im}(f)$. We denote by $\text{Hom}(X, Y)$ the set of all homomorphisms of $Q$-algebras from $X$ to $Y$. 
**Proposition 3.16.** Suppose that \( f : X \rightarrow X' \) is a homomorphism of \( Q \)-algebras. Then

1. \( f(0) = 0' \),
2. \( f \) is isotone, that is, if \( x * y = 0, x, y \in X \), then \( f(x) * f(y) = 0' \).

**Proof.** Since \( f(0) = f(0 * 0) = f(0) * f(0) = 0' \), (1) holds. If \( x, y \in X \) and \( x \leq y \), that is, \( x * y = 0 \), then by (1), \( f(x) * f(y) = f(x * y) = f(0) = 0' \). Hence \( f(x) \leq f(y) \), proving (2).

**Theorem 3.17.** Let \( (X; *, 0) \) and \( (X; *, 0') \) be \( Q \)-algebras and let \( B \) be an ideal of \( Y \). Then for any \( f \in \text{Hom}(X, Y) \), \( f^{-1}(B) \) is an ideal of \( X \).

**Proof.** By Proposition 3.16(1), \( 0 \in f^{-1}(B) \). Assume that \( x * y \in f^{-1}(B) \) and \( y \in f^{-1}(B) \). Then \( f(x) * f(y) = f(x * y) \in B \). It follows from the fact that \( B \) is an ideal of \( Y \) that \( f(x) \in B \), that is, \( x \in f^{-1}(B) \). This means that \( f^{-1}(B) \) is an ideal of \( X \). The proof is complete.

Since \( \{0'\} \) is an ideal of \( X' \), \( \text{Ker}(f) = f^{-1}(\{0'\}) \) for any \( f \in \text{Hom}(X, Y) \). Hence we obtain the following corollary.

**Corollary 3.18.** The kernel \( \text{Ker}(f) \) is an ideal of \( X \).

4. The quadratic \( Q \)-algebras. Let \( X \) be a field with \(|X| \geq 3\). An algebra \( (X; *) \) is said to be quadratic if \( x * y \) is defined by \( x * y := a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6 \), where \( a_1, \ldots, a_6 \in X \), for any \( x, y \in X \). A quadratic algebra \( (X; *) \) is said to be quadratic \( Q \)-algebra (resp., \( QS \)-algebra) if it satisfies conditions (I), (II), and (III) (resp., (IV)).

**Theorem 4.1.** Let \( X \) be a field with \(|X| \geq 3\). Then every quadratic \( Q \)-algebra \( (X; *, e) \), \( e \in X \), has the form \( x * y = x - y + e \) where \( x, y \in X \).

**Proof.** Define

\[
x * y := Ax^2 + Bxy + Cy^2 + Dx + Ey + F. \tag{4.1}
\]

Consider (I).

\[
e = x * x = (A + B + C)x^2 + (D + E)x + F. \tag{4.2}
\]

Let \( x := 0 \) in (4.2). Then we obtain \( F = e \). Hence (4.1) turns out to be

\[
x * y = Ax^2 + Bxy + Cy^2 + Dx + Ey + e. \tag{4.3}
\]

If \( y := x \) in (4.3), then

\[
e = x * x = (A + B + C)x^2 + (D + E)x + e, \tag{4.4}
\]

for any \( x \in X \), and hence we obtain \( A + B + C = 0 = D + E \), that is, \( E = -D \) and \( B = -A - C \). Hence (4.3) turns out to be

\[
x * y = (x - y)(Ax - Cy + D) + e. \tag{4.5}
\]

Let \( y := e \) in (4.5). Then by (II) we have

\[
x = x * e = (x - e)(Ax - Ce + D) + e, \tag{4.6}
\]
that is, \((Ax - Ce + D - 1)(x - e) = 0\). Since \(X\) is a field, either \(x - e = 0\) or \(Ax - Ce + D - 1 = 0\). Since \(|X| \geq 3\), we have \(Ax - Ce + D - 1 = 0\), for any \(x \neq e\) in \(X\). This means that \(A = 0, 1 - D + Ce = 0\). Thus (4.5) turns out to be
\[
x * y = (x - y) + C(x - y)(e - y) + e.
\] (4.7)

To satisfy condition (III) we consider \((x * y) * z\) and \((x * z) * y\).
\[
(x * y) * z = (x * y - z) + C(x * y - z)(e - z) + e
\]
\[
= (x - y - z) + C(x - y)(e - z) + 2e
\]
\[
+ C[(x - y) + C(x - y)(e - y) + (e - z)](e - z)
\] (4.8)
\[
= (x - y - z) + C(x - y)(2e - y - z) + 2e
\]
\[
+ C^2(x - y)(e - y)(e - z) + C(e - z)^2.
\]

Interchange \(y\) with \(z\) in (4.8). Then
\[
(x * z) * y = (x - z - y) + C(x - z)(2e - z - y) + 2e
\]
\[
+ C^2(x - z)(e - y)(e - z) + C(e - y)^2.
\] (4.9)

By (4.8) and (4.9) we obtain
\[
0 = (x * y) * z - (x * z) * y = C^2(e - y)(e - z)(z - y).
\] (4.10)

Since \(X\) is a field with \(|X| \geq 3\), we obtain \(C = 0\). This means that every quadratic \(Q\)-algebra \((X; *, e)\), has the form \(x * y = x - y + e\) where \(x, y \in X\), completing the proof.

**Example 4.2.** Let \(\mathbb{R}\) be the set of all real numbers. Define \(x * y := x - y + \sqrt{2}\). Then \((\mathbb{R}; *, \sqrt{2})\) is a quadratic \(Q\)-algebra.

**Example 4.3.** Let \(\mathcal{K} := GF(p^n)\) be a Galois field. Define \(x * y := x - y + e, e \in \mathcal{K}\). Then \((\mathcal{K}; *, e)\) is a quadratic \(Q\)-algebra.

**Theorem 4.4.** Let \(X\) be a field with \(|X| \geq 3\). Then every quadratic \(Q\)-algebra on \(X\) is a (quadratic) \(QS\)-algebra.

**Proof.** Let \((X; *, e)\) be a quadratic \(Q\)-algebra. Then \(x * y = x - y + e\) for any \(x, y \in X\), and hence
\[
(x * y) * (x * z) = (x - y + e) * (x - z + e)
\]
\[
= (x - y + e) - (x - z + e) + e
\] (4.11)
\[
= z - y + e = z * y,
\]
completing the proof.

**Remark 4.5.** Usually a nonquadratic \(Q\)-algebra need not be a \(QS\)-algebra. See the following example.
EXAMPLE 4.6. Consider the $Q$-algebra $(X;\ast,0)$ in Example 2.2. This algebra is not a $QS$-algebra, since $(3 \ast 1) \ast (3 \ast 2) = 3 \neq 0 = 2 \ast 1$.

COROLLARY 4.7. Let $X$ be a field with $|X| \geq 3$. Then every quadratic $Q$-algebra on $X$ is a BCI-algebra.

PROOF. It is an immediate consequences of Theorems 2.5 and 4.4. \hfill \Box

THEOREM 4.8. Let $X$ be a field with $|X| \geq 3$. Then every quadratic $Q$-algebra $(X;\ast,e)$ is $p$-semisimple. Furthermore, if $\text{char}(X) \neq 2$, then $G(X) = B(X)$.

PROOF. Notice that $B(X) = \{x \in X \mid e \ast x = e\} = \{x \in X \mid e - x + e = e\} = \{x \in X \mid e - x = 0\} = \{e\}$, that is, $(X;\ast,e)$ is $p$-semisimple. Also, if $\text{char}(X) \neq 2$, then 2 is invertible in $X$ and $G(X) = \{x \in X \mid e \ast x = x\} = \{x \in X \mid e - x + e = x\} = \{x \in X \mid 2e = 2x\} = \{x \in X \mid e = e\}$. Of course, if $\text{char}(X) = 2$, then $2e = 2x = 0$ for all $x \in X$, whence $G(X) = X$. \hfill \Box

This shows that there is a large class of examples of $p$-semisimple $QS$-algebras obtained as quadratic $Q$-algebras.

THEOREM 4.9. Let $X$ be a field with $|X| \geq 3$. Then every quadratic $Q$-algebra on $X$ is isomorphic to every other such algebra defined on $X$.

PROOF. Let $x \ast y := x - y + e_1$ and $x \ast' y := x - y + e_2$, where $e_1,e_2 \in X$. Let $\pi(x) := x + (e_2 - e_1)$, for all $x \in X$. Then $\pi(x \ast y) = [(x - y) + e_1] + (e_2 - e_1) = (x - y) + e_2 = (x + (e_2 - e_1)) + (y + (e_2 - e_1)) + e_2 = \pi(x) \ast' \pi(y)$, whence the fact that $\pi^{-1}(x) = x + (e_1 - e_2)$ yields the conclusion that $\pi$ is an isomorphism of $Q$-algebras. \hfill \Box

THEOREM 4.10. Let $X$ be a field with $|X| \geq 3$. Then every quadratic $Q$-algebra $(X;\ast,e)$ determines the abelian group $(X,\ast)$ via the definition $x \ast y = x \ast (e - y)$.

PROOF. Note that $x \ast (e - y) = x - (e - y) + e = x + y$ returns the additive operation of the field $X$, which is an abelian group. \hfill \Box

Not every quadratic $Q$-algebra $(X;\ast,e)$, $e \in X$, on a field $X$ with $|X| \geq 3$ need be a $BCK$-algebra, since $((x \ast y) \ast (x \ast z)) \ast (z \ast y) = e + (y - z) \neq e$ in general.

PROBLEM 4.11. Construct a cubic $Q$-algebra which is not quadratic. Verify that among such cubic $Q$-algebras there are examples which are not $QS$-algebras. Furthermore, the question whether there are non-$p$-semisimple cubic $Q$-algebras is also of interest.

REFERENCES


JOSEPH NEGgers: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA, TUSCALOOSA, AL 35487-0350, USA

E-mail address: jneggers@gp.as.ua.edu

Sun Shin Ahn: DEPARTMENT OF MATHEMATICS EDUCATION, DONGGUK UNIVERSITY, SEOUL 100-715, KOREA

E-mail address: sunshine@dgu.ac.kr

Hee Sik Kim: DEPARTMENT OF MATHEMATICS, HANYANG NATIONAL UNIVERSITY, SEOUL 133-791, KOREA

E-mail address: heekim@hanyang.ac.kr
Submit your manuscripts at http://www.hindawi.com