REGULAR-UNIFORM CONVERGENCE AND THE OPEN-OPEN TOPOLOGY

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ABSTRACT. In 1994, Bânzaru introduced the concept of regular-uniform, or r-uniform, convergence on a family of functions. We discuss the relationship between this topology and the open-open topology, which was described in 1993 by Porter, on various collections of functions.

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1. Introduction. In [1], Bânzaru introduced the concept of regular-uniform, or r-uniform, convergence on a family of functions \( F \subset Y^X \) and proved a number of facts about the topological space \((F, T_r)\) where \( T_r \) is the topology induced by this convergence. Porter introduced the open-open topology [5] in 1993 and proved that on families of self-homeomorphisms on \( X \) that the open-open topology is equivalent to the topology of Pervin quasi-uniform convergence [3]; this in fact is true on \( C(X, Y) \), the collection of all continuous functions from \( X \) to \( Y \). We shall show that the topology of r-uniform convergence on any subfamily \( F \) of the class of all continuous functions on \( X \) into \( Y \) is equivalent to the open-open topology [5], \( T_{oo} \), on \( F \) and hence, equivalent to the topology of Pervin quasi-uniform convergence on \( F \).

Throughout this paper let \((X, T)\) and \((Y, T')\) be topological spaces. We will use \( Y^X \) to mean the collection of all functions from \( X \) into \( Y \) while \( C(X, Y) \) will represent the collection of all continuous functions from \( X \) into \( Y \), and \( H(X) \) is the collection of all self-homeomorphisms on \( X \).

2. Preliminaries. A net of functions \( \{f_\alpha: (X, T) \rightarrow (Y, T')\}_{\alpha \in I} \) converges r-uniformly (or regular uniformly) to \( f \in Y^X \) [1] if and only if for any \( O \in T' \) such that \( f^{-1}(O) \neq \phi \), there exists \( i_0 \in I = [0, 1] \) such that \( f_i(x) \in O \) for all \( i \in I \) with \( i \geq i_0 \) and for all \( x \in f^{-1}(O) \). This convergence defines a topology on \( F \) called the topology of r-uniform or regular uniform convergence.

In the same paper, Bânzaru also defined a topology, \( T_r \), on \( F \subset Y^X \) as follows: let \( f \in F \) and \( O \in T' \). Set

\[
S(f; O) = \{g \in F : g(f^{-1}(O)) \subset O\},
\]

then \( S = \{S(f; O) : f \in F \text{ and } O \in T'\} \) is a subbasis for a topology \( T_r \) on \( F \). Bânzaru then proved that this topology \( T_r \) on \( F \) is actually equivalent to the topology of r-uniform convergence on \( F \).
Now let \( O \in T \) and \( U \in T' \) and define

\[
(U,V) = \{ h \in F : h(O) \subseteq U \}.
\] (2.2)

Then \( S_{oo} = \{(O,U) : O \in T \text{ and } U \in T'\} \) is a subbasis for the open-open topology, \( T_{oo} \), [5] on \( F \).

In addition, the set \( S_{co} = \{(C,U) \subseteq F : C \text{ is compact in } X \text{ and } U \text{ is open in } Y\} \) is a subbasis for the well-known compact-open topology, \( T_{co} \), on \( F \).

Let \( X \) be a nonempty set and let \( Q \) be a collection of subsets of \( X \times X \) such that

1. for all \( U \in Q \), \( \Delta = \{(x,x) \in X \times X : x \in X\} \subseteq U \),
2. for all \( U \in Q \), if \( U \subseteq V \) then \( V \in Q \),
3. for all \( U,V \in Q \), \( U \cap V \in Q \), and
4. for all \( U \in Q \), there exists some \( W \in Q \) such that \( W \circ W \subseteq U \) where \( W \circ W = \{(p,q) \in X \times X : \text{there exists some } r \in X \text{ with } (p,r),(r,q) \in W\} \) then \( Q \) is a quasi-uniformity on \( X \).

A quasi-uniformity, \( Q \), on \( X \) induces a topology, \( T_Q \), on \( X \), where for each \( x \in X \), the set \( \{U[x] : U \in Q\} \) is a neighborhood system at \( x \) where \( U[x] \) is defined by \( U[x] = \{y \in X : (x,y) \in U\} \).

A family, \( S \) of subsets of \( X \times X \) which satisfies

(i) for all \( R \in S \), \( \Delta \subseteq R \), and
(ii) for all \( R \in S \), there exists \( T \in S \) such that \( T \circ T \subseteq R \), is a subbasis for a quasi-uniformity, \( Q \), on \( X \). This subbasis \( S \) generates a basis, \( B \), for the quasi-uniformity, \( Q \), where \( B \) is the collection of all finite intersections of elements of \( S \). The basis, \( B \), generates the quasi-uniformity \( Q = \{ U \subseteq X \times X : \hat{B} \subseteq U \text{ for some } \hat{B} \in B\} \).

For a more thorough background on quasi-uniform spaces, see [2].

In 1962, Pervin [4] constructed a specific quasi-uniformity which induces a compatible topology for a given topological space. His construction is as follows: Let \( (X,T) \) be a topological space. For \( O \in T \) define

\[
S_O = (O \times O) \cup ((X \setminus O) \times X).
\] (2.3)

One can show that for \( O \in T \), \( S_O \circ S_O = S_O \) and \( \Delta \subseteq S_O \), hence, the collection \( \{S_O : O \in T\} \) is a subbasis for a quasi-uniformity, \( P \), on \( X \), called the Pervin quasi-uniformity.

Let \( Q \) be a compatible quasi-uniformity for \( (X,T) \) and let \( F \subseteq C(X,Y) \). For \( U \in Q \), define the set

\[
W(U) = \{(f,g) \subseteq F \times F : (f(x),g(x)) \subseteq U \text{ for all } x \in X\}.
\] (2.4)

Then the collection \( B = \{W(U) : U \in Q\} \) is a basis for a quasi-uniformity, \( Q^* \), on \( F \), called the quasi-uniformity of quasi-uniform convergence with respect to \( Q \) [3]. The topology, \( T_{Q^*} \), induced by \( Q^* \) on \( F \), is called the topology of quasi-uniform convergence with respect to \( Q \). If \( Q \) is the Pervin quasi-uniformity, \( P \), then \( T_{P^*} \) is called the topology of Pervin quasi-uniform convergence.
3. The topologies. We first extend, to subsets of $C(X,Y)$, the result from [5] that the open-open topology is equivalent to the topology of Pervin quasi-uniform convergence on a subgroup $G$ of $H(X)$.

**Theorem 3.1.** Let $F \subset C(X,Y)$. The open-open topology, $T_{oo}$, is equivalent to the topology of Pervin quasi-uniform convergence, $T_{Perv}$, on $F$.

**Proof.** Assume $F \subset C(X,Y)$. Let $(O,U)$ be a subbasic open set in $T_{oo}$ and let $f \in F$. Then $f(O) \subset U$. So $f \in W(S(U))[f]$ where

$$W(S(U))[f] = \{g \in F : (f(x),g(x)) \in S(U) = U \times U \cup (X \setminus U) \times X, \forall x \in X\}. \quad (3.1)$$

Hence, if $g \in W(S(U))[f]$ and $x \in O$, then $f(x) \in U$ so $g(x) \in U$. Thus, $g \in (O,U)$ and $W(S(U))[f] \subset (O,U)$. Therefore, $T_{oo} \subset T_{Perv}$.

Now let $V \in T_{Perv}$ and $f \in V$. Then there exists $U \in P$ such that $f \in W(U)[f] \subset V$. Since $U \in P$, there exists some finite collection, $\{U_i : i = 1,2,\ldots,n\} \subset T$ such that $\cap_{i=1}^n S(U_i) \subset U$. Define $A = \cap_{i=1}^n (f^{-1}(U_i),U_i)$. Then $A$ is an open set in $T_{oo}$ and $f \in A$. Assume $g \in A$ and let $x \in X$. If $f(x) \in U_j$ for some $j \in \{1,2,\ldots,n\}$, then $x \in f^{-1}(U_j)$. Then, since $g \in A$, $g(x) \in U_j$, hence, $(f(x),g(x)) \in U_j \times U_j \subset S(U_j)$. If $f(x) \notin U_j$ for some $j \in \{1,2,\ldots,n\}$, then $(f(x),g(x)) \in (X - U_j) \times X \subset S(U_j)$. Thus, $g \in W(\cap_{i=1}^n S(U_i))[f] \subset W(U)[f] \subset V$ so that $A \subset V$. Therefore, $T_{oo} = T_{Perv}$ on $F$. \hfill \Box

Next we show that the regular-uniform topology is equivalent to the open-open topology on any subset, $F$, of $C(X,Y)$, and hence, also to the topology of Pervin quasi-uniform convergence on $F$.

**Theorem 3.2.** For $F \subset C(X,Y)$, $T_{oo} = T_{r}$ on $F$.

**Proof.** Note that a subbasic open set in $T_{r}$, $S(f;O) = \{g \in F : g(f^{-1}(O)) \subset O\}$ is equal to $(f^{-1}(O),O)$. Hence, if $f^{-1}(O)$ is open in $X$, which is the case when $f$ is continuous, $S(f;O)$ is a subbasic open set in $T_{oo}$. Therefore, $T_{r} \subset T_{oo}$.

Now let $(O,U)$ be a subbasic open set in $T_{oo}$ and let $f \in (O,U)$. Then $f(O) \subset U$ which implies that $O \subset f^{-1} \circ f(O) \subset f^{-1}(U)$. Since $f \circ f^{-1}(U) = U$, $f \in (f^{-1}(U),U) = S(f;U) \subset T_{r}$. If $g \in (f^{-1}(U),U)$, then $g(f^{-1}(U)) \subset U$. If $x \in O$, then $x \in f^{-1}(U)$ so that $g(x) \in U$ giving us that $g \in (O,U)$, whence $T_{oo} \subset T_{r}$ and we are done. \hfill \Box

While it is always true that $T_{oo} \subset T_{r}$ on $F \subset Y^{X}$, it is not necessarily true that $T_{r} = T_{oo}$ for $F \subset Y^{X}$ as the following example shows.

**Example 3.3.** Define the sets $X = \{1,2,3\}$, $T = \{\{1\},\phi,X\}$, $Y = \{1,2,3,4\}$, $T' = \{\{1,2\},\{3,4\},\phi,Y\}$ and $F = \{f_1,f_2,f_3,f_4\}$ which are given in Table 3.1. Then $T_{oo} = \{\phi,F,\{f_1,f_2,f_3\},\{f_4\}\}$. But $S(f_3;\{3,4\}) = \{f_3\} \notin T_{oo}$. In fact, $T_{r}$ is the discrete topology on $F$.

Bânsaru proved that for any $F \subset Y^{X}$, the compact-open topology, $T_{co}$, is coarser than $T_{r}$. However, although $T_{co} \subset T_{oo}$ on $F$ when $F \subset C(X,Y)$, it is not necessarily true that $T_{co} \subset T_{oo}$ for $F \subset Y^{X}$. Consider Example 3.3 again. We have that $\{\{2\},\{3,4\}\}$ is in $T_{co}$ and equals $\{f_3\}$, but $\{f_3\} \notin T_{oo}$. In this example, the compact-open topology on $F$ is also the discrete topology and thus equals the regular-uniform topology on $F$. 


Another fact that has been proved in [1] about the regular-uniform topology is that if the topology for $Y$ is regular, then $(C(X,Y), T_r)$ is closed in $(Y^X, T_r)$. However, this is not true when $Y^X$ is given the open-open topology; that is, let $(X,T)$ and $(Y,T')$ be topological spaces such that $(Y,T')$ is regular. Then $(C(X,Y), T_r)$, which is the same as $(C(X,Y), T_{oo})$, is not necessarily closed in $(Y^X, T_{oo})$. The following example illustrates this.

**Example 3.4.** Let $X = \{1, 2\}$, $T = \{\emptyset, X, \{1\}\}$, $Y = \{1, 2, 3\}$, and $T' = \{\emptyset, Y, \{1\}, \{2, 3\}\}$. The collection $Y^X$ is given in Table 3.2. Note that $T'$ is a partition topology and is thus regular. Also note that $f_1^{-1}(\{2, 3\}) = \{2\}$ and so $f_1$ is not continuous. The only open sets in $(Y^X, T_{oo})$ that contain $f_1$ are $(\emptyset, Y) = Y^X$ and $(\{1\}, \{1\}) = \{f_1, f_2, f_3\}$. Both of these sets contain the function $f_2$ which is continuous. Thus, $C(X,Y)$ is not closed in $(Y^X, T_{oo})$, even though $(Y, T')$ is regular.

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**References**


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