AN EXTENSION OF A THEOREM OF SAHAB, KHAN, AND SESSA

A. R. KHAN and N. HUSSAIN

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ABSTRACT. A fixed point theorem of Fisher and Sessa is generalized to locally convex spaces and the new result is applied to extend a recent theorem on invariant approximation of Sahab, Khan, and Sessa.

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1. Introduction and terminology. In 1988, Mukherjee and Verma [10] obtained the following generalization of a theorem of Fisher and Sessa [4].

THEOREM 1.1. *Let T and I be two weakly commuting mappings of a closed convex subset C of a Banach space X into itself satisfying the inequality*

$$\|Tx - Ty\| \leq a\|Ix - Iy\| + (1 - a) \max \{\|Tx - Ix\|, \|Ty - Iy\|\}, \quad (1.1)$$

for all $x, y \in C$, where $a \in (0, 1)$. If I is affine and nonexpansive in C and if $T(C) \subseteq I(C)$, then T and I have a unique common fixed point in C .

In this note, we first prove that Theorem 1.1 can appreciably be extended to the setup of a Hausdorff locally convex space. An application of new result is presented to best approximation theory; our work extends earlier results of Brosowski [3], Sahab et al. [12], Singh [14] and many others.

In the sequel, (E, τ) will be a Hausdorff locally convex topological vector space. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is said to be an associated family of seminorms for τ if the family $\{\gamma U : \gamma > 0\}$, where $U = \bigcap_{i=1}^n U_{\alpha_i}$ and $U_{\alpha_i} = \{x : p_{\alpha_i}(x) < 1\}$, forms a base of neighbourhoods of zero for τ . A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is called an augmented associated family for τ if $\{p_\alpha : \alpha \in I\}$ is an associated family with the property that the seminorm $\max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in I\}$ for any $\alpha, \beta \in I$. The associated and augmented associated families of seminorms will be denoted by $A(\tau)$ and $A^*(\tau)$, respectively. It is well known that given a locally convex space (E, τ) , there always exists a family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E such that $\{p_\alpha : \alpha \in I\} = A^*(\tau)$ (see [9, page 203]).

The following construction will be crucial. Suppose that M is a τ -bounded subset of E . For this set M we can select a number $\lambda_\alpha > 0$ for each $\alpha \in I$ such that $M \subset \lambda_\alpha U_\alpha$ where $U_\alpha = \{x : p_\alpha(x) \leq 1\}$. Clearly $B = \bigcap_\alpha \lambda_\alpha U_\alpha$ is τ -bounded, τ -closed, absolutely convex and contains M . The linear span E_B of B in E is $\bigcup_{n=1}^\infty nB$. The Minkowski functional of B is a norm $\|\cdot\|_B$ on E_B . Thus $(E_B, \|\cdot\|_B)$ is a normed space with B as its closed unit ball and $\sup_\alpha p_\alpha(x/\lambda_\alpha) = \|x\|_B$ for each $x \in E_B$.

Following Sessa [13], we say, two selfmaps I and T of a locally convex space (E, τ) are weakly commuting if and only if

$$p_\alpha(ITx - TIx) \leq p_\alpha(Ix - Tx), \quad (1.2)$$

for each $x \in E$ and $p_\alpha \in A^*(\tau)$. Clearly, commuting maps are weakly commuting but not conversely in general (see [10, 13]). A mapping $T : E \rightarrow E$ is said to be nonexpansive on E if $p_\alpha(Tx - Ty) \leq p_\alpha(x - y)$ for all x, y in E and $p_\alpha \in A^*(\tau)$. The set of fixed points of T on E is denoted by $F(T)$. If $u \in E$, $M \subseteq E$, then for $0 < a \leq 1$, we define the set D_a of best (M, a) -approximants to u as follows:

$$D_a = \{y \in M : ap_\alpha(y - u) = d_{p_\alpha}(u, M), \forall p_\alpha \in A^*(\tau)\}, \quad (1.3)$$

where

$$d_{p_\alpha}(u, M) = \inf \{p_\alpha(x - u) : x \in M\}. \quad (1.4)$$

Let D denote the set of best approximations to u . For $a = 1$, our definition reduces to the set D of best M -approximants to u . A mapping $T : M \rightarrow E$ is called demiclosed at 0 if whenever $\{x_n\}$ converges weakly to x and $\{Tx_n\}$ converges to 0, we have $Tx = 0$.

2. Results

LEMMA 2.1. *Let T and I be weakly commuting selfmaps of a τ -bounded subset M of a Hausdorff locally convex space (E, τ) . Then T and I are weakly commuting on M with respect to $\|\cdot\|_B$.*

PROOF. By hypothesis for any $x \in M$,

$$p_\alpha(ITx - TIx) \leq p_\alpha(Ix - Tx), \quad \text{for each } p_\alpha \in A^*(\tau). \quad (2.1)$$

Taking supremum on both sides, we get

$$\sup_\alpha p_\alpha \left(\frac{ITx - TIx}{\lambda_\alpha} \right) \leq \sup_\alpha p_\alpha \left(\frac{Ix - Tx}{\lambda_\alpha} \right), \quad (2.2)$$

$$\|ITx - TIx\|_B \leq \|Ix - Tx\|_B \quad \text{as desired.} \quad \square$$

Note that if I is nonexpansive on a τ -bounded subset M of E , then I is also nonexpansive with respect to $\|\cdot\|_B$ (cf. [8, 15]).

We use a technique of Tarafdar [15] to obtain the following common fixed point theorem which generalizes Theorem 1.1 and the main result of Fisher and Sessa [4].

THEOREM 2.2. *Let M be a nonempty τ -bounded, τ -complete, and convex subset of a Hausdorff locally convex space (E, τ) and T, I two weakly commuting selfmaps of M satisfying the inequality*

$$p_\alpha(Tx - Ty) \leq ap_\alpha(Ix - Iy) + (1 - a) \max \{p_\alpha(Tx - Ix), p_\alpha(Ty - Iy)\}, \quad (2.3)$$

for all $x, y \in M$ and for all $p_\alpha \in A^(\tau)$ and for some $a \in (0, 1)$. If I is affine and nonexpansive on M and $T(M) \subseteq I(M)$, then T and I have a unique common fixed point.*

PROOF. Since M is τ -complete, it follows that $(E_B, \|\cdot\|_B)$, is a Banach space and M is complete in it. By Lemma 2.1, T and I are $\|\cdot\|_B$ -weakly commuting maps of M . From (2.3) we obtain for $x, y \in M$,

$$\begin{aligned} \sup_{\alpha} p_{\alpha}\left(\frac{Tx - Ty}{\lambda_{\alpha}}\right) &\leq a \sup_{\alpha} p_{\alpha}\left(\frac{Ix - Iy}{\lambda_{\alpha}}\right) \\ &\quad + (1-a) \max \left\{ \sup_{\alpha} p_{\alpha}\left(\frac{Tx - Ix}{\lambda_{\alpha}}\right), \sup_{\alpha} p_{\alpha}\left(\frac{Ty - Iy}{\lambda_{\alpha}}\right) \right\}. \end{aligned} \quad (2.4)$$

Thus

$$\|Tx - Ty\|_B \leq a \|Ix - Iy\|_B + (1-a) \max \{ \|Tx - Ix\|_B, \|Ty - Iy\|_B \}. \quad (2.5)$$

It can be shown easily that I is $\|\cdot\|_B$ -nonexpansive on M . A comparison of our hypothesis with that of Theorem 1.1 tells that we can apply Theorem 1.1 to M as a subset of $(E_B, \|\cdot\|_B)$ to conclude that there exists a unique $a \in M$ such that $a = Ia = Ta$. $\square$

An application of Theorem 2.2 establishes the following result in best approximation theory.

THEOREM 2.3. *Let T and I be selfmaps of a Hausdorff locally convex space (E, τ) and M a subset of E such that $T(\partial M) \subseteq M$, where ∂M denotes boundary of M and $u \in F(T) \cap F(I)$. Suppose that T and I satisfy (2.3) for all x, y in $D'_a = D_a \cup \{u\}$ and I is nonexpansive and affine on D_a . For each $p_{\alpha} \in A^*(\tau)$,*

$$p_{\alpha}(TIx - ITx) \leq \frac{1}{k} p_{\alpha}((kTx + (1-k)q) - Ix), \quad (2.6)$$

for all $k \in (0, 1)$, $x \in D_a$ and for some $q \in D_a$. If D_a is nonempty convex, $q \in F(I)$ and $I(D_a) = D_a$, then I and T have a common fixed point in D_a provided one of the following conditions holds:

- (i) D_a is τ -compact.
- (ii) D_a is weakly compact in (E, τ) , I is weakly continuous and $I - T$ is demiclosed at 0.

PROOF. Let $y \in D_a$. Then $Iy \in D_a$, since $I(D_a) = D_a$. Further, if $y \in \partial M$, then $Iy \in M$ for $T(\partial M) \subseteq M$. From (2.3), it follows that for each $p_{\alpha} \in A^*(\tau)$,

$$\begin{aligned} p_{\alpha}(Ty - u) &= p_{\alpha}(Ty - Tu) \\ &\leq ap_{\alpha}(Iy - Iu) + (1-a) \max \{ p_{\alpha}(Ty - Iy), p_{\alpha}(Tu - Iu) \} \\ &\leq ap_{\alpha}(Iy - u) + (1-a) (p_{\alpha}(Ty - u) + p_{\alpha}(Iy - u)). \end{aligned} \quad (2.7)$$

So we have, $ap_{\alpha}(Ty - u) \leq p_{\alpha}(Iy - u)$ for all $p_{\alpha} \in A^*(\tau)$. Hence $Ty \in D_a$ which implies that T maps D_a into itself.

Let $\{k_n\}$ be a monotonically nondecreasing sequence of real numbers such that $0 < k_n < 1$ and $\limsup k_n = 1$. Define for each $n \in \mathbb{N}$, a mapping $T_n : D_a \rightarrow D_a$ by

$$T_n(x) = k_n Tx + (1 - k_n)q. \quad (2.8)$$

It is possible to define such a mapping T_n for each $n \in \mathbb{N}$, since D_a is convex and

$q \in D_a$. The map I is affine so we have

$$T_n Ix = k_n T Ix + (1 - k_n)q, \quad I T_n x = k_n I T x + (1 - k_n)q. \quad (2.9)$$

From (2.6), it follows that

$$\begin{aligned} p_\alpha(T_n Ix - I T_n x) &= k_n p_\alpha(T Ix - I T x) \\ &\leq k_n \left(\left(\frac{1}{k_n} \right) p_\alpha(k_n T x + (1 - k_n)q) - Ix \right) \\ &= p_\alpha(T_n x - Ix), \quad \forall x \in D_a, p_\alpha \in A^*(\tau). \end{aligned} \quad (2.10)$$

Thus I and T_n are weakly commuting on D_a for each n and $T_n(D_a) \subseteq D_a = I(D_a)$. For all $x, y \in D_a$, $p_\alpha \in A^*(\tau)$ and for all $j \geq n$, (n fixed), we obtain from (2.3),

$$\begin{aligned} p_\alpha(T_n x - T_n y) &= k_n p_\alpha(T x - T y) \leq k_j p_\alpha(T x - T y) \\ &\leq p_\alpha(T x - T y) \\ &\leq a p_\alpha(Ix - Iy) + (1 - a) \max \{ p_\alpha(T x - Ix), p_\alpha(T y - Iy) \} \\ &\leq a p_\alpha(Ix - Iy) + (1 - a) \max \{ p_\alpha(T x - T_n x) + p_\alpha(T_n x - Ix), \\ &\quad p_\alpha(T y - T_n y) + p_\alpha(T_n y - Iy) \} \\ &\leq a p_\alpha(Ix - Iy) + (1 - a) \max \{ (1 - k_n) p_\alpha(T x - q) + p_\alpha(T_n x - Ix), \\ &\quad (1 - k_n) p_\alpha(T y - q) + p_\alpha(T_n y - Iy) \}. \end{aligned} \quad (2.11)$$

Hence for all $j \geq n$, we have

$$\begin{aligned} p_\alpha(T_n x - T_n y) &\leq a p_\alpha(Ix - Iy) \\ &\quad + (1 - a) \max \{ (1 - k_j) p_\alpha(T x - q) + p_\alpha(T_n x - Ix), \\ &\quad (1 - k_j) p_\alpha(T y - q) + p_\alpha(T_n y - Iy) \}. \end{aligned} \quad (2.12)$$

As $\lim k_j = 1$, from (2.12), for every $n \in \mathbb{N}$, we have

$$\begin{aligned} p_\alpha(T_n x - T_n y) &= \lim_j p_\alpha(T_n x - T_n y) \\ &\leq \lim_j \{ a p_\alpha(Ix - Iy) + (1 - a) \\ &\quad \times \max \{ (1 - k_j) p_\alpha(T x - q) + p_\alpha(T_n x - Ix), \\ &\quad (1 - k_j) p_\alpha(T y - q) + p_\alpha(T_n y - Iy) \} \}. \end{aligned} \quad (2.13)$$

This implies that for every $n \in \mathbb{N}$,

$$p_\alpha(T_n x - T_n y) \leq a p_\alpha(Ix - Iy) + (1 - a) \max \{ p_\alpha(T_n x - Ix), p_\alpha(T_n y - Iy) \}, \quad (2.14)$$

for all $x, y \in D_a$ and for all $p_\alpha \in A^*(\tau)$.

(i) D_a being τ -compact is τ -bounded and τ -complete. Thus by Theorem 2.2, for every $n \in \mathbb{N}$, T_n and I have unique common fixed point x_n in D_a . Now the τ -compactness

of D_a ensures that $\{x_n\}$ has a convergent subsequence $\{x_{n_j}\}$ which converges to a point $x_0 \in D_a$. Since

$$x_{n_j} = T_{n_j}x_{n_j} = k_{n_j}Tx_{n_j} + (1 - k_{n_j})q \quad (2.15)$$

and T is continuous, so we have, as $j \rightarrow \infty$, $Tx_0 = x_0$. The continuity of I implies that

$$Ix_0 = I\left(\lim_j x_{n_j}\right) = \lim_j I(x_{n_j}) = \lim_j x_{n_j} = x_0. \quad (2.16)$$

(ii) Weakly compact sets in (E, τ) are τ -bounded and τ -complete so again by [Theorem 2.2](#), T_n and I have a common fixed point x_n in D_a for each n . The set D_a is weakly compact so there is a subsequence $\{x_j\}$ of $\{x_n\}$ converging weakly to some $y \in D_a$. The map I being weakly continuous gives that $Iy = y$. Now

$$x_j = I(x_j) = T_j(x_j) = k_jTx_j + (1 - k_j)q \quad (2.17)$$

implies that $Ix_j - Tx_j = (1 - k_j)[q - Tx_j] \rightarrow 0$ as $j \rightarrow \infty$. The demiclosedness of $I - T$ at 0 implies that $(I - T)(y) = 0$. Hence $Iy = Ty = y$. $\square$

EXAMPLE 2.4 (cf. MR.89h:54030). Let $M = [1, \infty)$ and d be the absolute value metric on M . Define f and g on M by $fx = 1 + x$, $g(x) = 1 + 2x$. As $d(fgx, gfx) = 1 \leq x = d(fx, gx)$ for all x in M so f and g are weakly commuting but evidently there exists no sequence $\{x_n\}$ in M for which the condition of compatibility is satisfied (f and g are compatible (see [6]) if $d(fgx_n, gfx_n) \rightarrow 0$, as $n \rightarrow \infty$, for any sequence $\{x_n\}$ in M satisfying $\lim_n fx_n = \lim_n gx_n = t \in M$).

REMARKS 2.5. (i) In the light of [Example 2.4](#), the classes of weakly commuting and compatible maps are different and so the statement “weakly commuting maps are compatible” on page 977 in [6] is not valid. Hence [Theorem 2.3](#) cannot be implied by Theorem 5 of Pathak et al. [11] even in Banach space setting.

(ii) Commuting maps satisfy (2.6) so [Theorem 2.3\(i\)](#) is a proper generalization of the main results of Sahab et al. [12] and Singh [14].

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A. R. KHAN: DEPARTMENT OF MATHEMATICAL SCIENCES, KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS, DHAHRAN 31261, SAUDI ARABIA

E-mail address: arahim@kfupm.edu.sa

N. HUSSAIN: CENTER FOR ADVANCED STUDIES IN PURE AND APPLIED MATHEMATICS, BAHAUDDIN ZAKARIYA UNIVERSITY, MULTAN 60800, PAKISTAN

E-mail address: mnawab2000@yahoo.com