SUFFICIENT CONDITIONS FOR MEROMORPHIC STARLIKENESS AND CLOSE-TO-CONVEXITY OF ORDER $\alpha$

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Abstract. The object of the present paper is to derive a property of certain meromorphic functions in the punctured unit disk. Our main theorem contains certain sufficient conditions for starlikeness and close-to-convexity of order $\alpha$ of meromorphic functions.

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1. Introduction. Let $\Sigma$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$$

(1.1)

which are analytic in the punctured unit disk $\mathbb{D} = \{z : 0 < |z| < 1\}$. A function $f \in \Sigma$ is said to be meromorphic starlike of order $\alpha$ if it satisfies

$$-\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathbb{U} = \mathbb{D} - \{0\})$$

(1.2)

for some $\alpha (0 \leq \alpha < 1)$. We denote by $\Sigma^*(\alpha)$ the class of all meromorphic starlike functions of order $\alpha$.

Let $\text{MC}(\alpha)$ be the subclass of $\Sigma$ consisting of functions $f$ which satisfy

$$-\Re \{z^2 f'(z)\} > \alpha \quad (z \in \mathbb{U})$$

(1.3)

for some $\alpha (0 \leq \alpha < 1)$. A function $f$ in $\text{MC}(\alpha)$ is meromorphic close-to-convex of order $\alpha$ in $\mathbb{D}$ (see [1]).

2. Main Result. In proving our main theorem, we need the following lemma due to Owa, Nunokawa, Saitoh, and Fukui [2].

Lemma 2.1. Let $p$ be analytic in $\mathbb{U}$ with $p(0) = 1$. Suppose that there exists a point $z_0 \in \mathbb{U}$ such that $\Re p(z) > 0 \ (|z| < |z_0|)$, $\Re p(z_0) = 0$, and $p(z) \neq 0$. Then we have $p(z) = ia(a \neq 0)$ and

$$\frac{z_0 p'(z_0)}{p(z_0)} = i \frac{k}{z} \left( a + \frac{1}{a} \right),$$

(2.1)

where $k$ is a real number with $k \geq 1$.

With the aid of Lemma 2.1, we derive the following theorem.
Then, applying Lemma 2.1, we have

This contradicts our assumption. Thus, we conclude that $\Re z^2 - \alpha f'(z) f''(z) > 0$ (\(z \in \mathbb{U}\)),

then

where $\alpha \leq 2$ and $(2(2 - \alpha) - 1)/2 \leq \beta < 2 - \alpha$.

**Proof.** We define the function $p$ in $\mathbb{U}$ by

with $y = 1/(1 + 2(2 - \alpha) - 2\beta)$. Then $p$ is analytic in $\mathbb{U}$ with $p(0) = 1$ and

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

Then, applying Lemma 2.1, we have $p(z) = ia(a \neq 0)$ and

It follows from this that

Therefore, we have

This contradicts our assumption. Thus, we conclude that $\Re p(z) > 0$ for all $z \in \mathbb{U}$, that is,

Putting $\beta = (2(2 - \alpha) - 1)/2$ in Theorem 2.2, we have

**Corollary 2.3.** If $f \in \Sigma$ satisfies $f(z), f'(z) \neq 0$ in $\mathbb{D}$ and

then

where $\alpha \leq 2$. 

Theorem 2.2. If $f \in \Sigma$ satisfies $f(z), f'(z) \neq 0$ in $\mathbb{D}$ and

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Putting $\beta = (2(2 - \alpha) - 1)/2$ in Theorem 2.2, we have

**Corollary 2.3.** If $f \in \Sigma$ satisfies $f(z), f'(z) \neq 0$ in $\mathbb{D}$ and

then

where $\alpha \leq 2$. 

Taking $\alpha = 1$ in Theorem 2.2, we have the following corollary.

**Corollary 2.4.** If $f \in \sum$ satisfies $f(z)f'(z) \neq 0$ in $\mathbb{D}$ and
\[
\Re \left\{ \frac{zf''(z)}{f'(z)} - \frac{zf'''(z)}{f''(z)} \right\} < 2 - \beta \quad (z \in \mathbb{D}),
\]
then
\[
-\Re \left\{ \frac{zf''(z)}{f'(z)} \right\} > \frac{1}{3 - 2\beta} \quad (z \in \mathbb{D}),
\]
that is, $f \in \sum^*(1/(3 - 2\beta))$, where $1/2 \leq \beta < 1$.

Further, letting $\alpha = 0$ in Theorem 2.2, we have the following corollary.

**Corollary 2.5.** If $f \in \sum$ satisfies $f(z)f'(z) \neq 0$ in $\mathbb{D}$ and
\[
-\Re \left\{ \frac{zf''(z)}{f'(z)} \right\} < 4 - \beta \quad (z \in \mathbb{D}),
\]
then
\[
-\Re \left\{ z^2 f''(z) \right\} > 5 - 2\beta \quad (z \in \mathbb{D}),
\]
that is, $f \in \text{MC}(1/(5 - 2\beta))$, where $3/2 \leq \beta < 2$. □

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**References**


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