A NEW INEQUALITY FOR A POLYNOMIAL

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Abstract. Let \( p(z) = a_0 + \sum_{j=t}^{n} a_j z^j \) be a polynomial of degree \( n \), having no zeros in \( |z| < k, k \geq 1 \), then it has been shown that for \( R > 1 \) and \( |z| = 1 \),

\[
|p(Rz) - p(z)| \leq \frac{(R^n - 1)}{(1 + kt + 1 + AtBt(kt + 1 + k^2t))}\max_{|z|=1} |p(z)| - \left(1 - \frac{1}{1 + k^n + 1 + AtBt(kt + 1 + k^2t)}\right)\frac{|z|^n}{m/k^n},
\]

where \( m = \min_{|z|=k} |p(z)| \), \( 1 \leq t < n \), \( At = \frac{(R^t - 1)}{(R^n - 1)} \), and \( Bt = |a_t/a_0| \). Our result generalizes and improves some well-known results.

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1. Introduction and statements of results. Let \( p(z) \) be a polynomial of degree \( n \),

\[
\begin{align*}
\max_{|z|=1} |p'(z)| & \leq n \max_{|z|=1} |p(z)|, \quad (1.1) \\
\max_{|z|=R>1} |p(z)| & \leq R^n \max_{|z|=1} |p(z)|. \quad (1.2)
\end{align*}
\]

Inequality (1.1) is a famous result known as Bernstein’s inequality (see [9]) where as inequality (1.2) is a simple consequence of maximum modulus principle [7]. Here in both inequalities (1.1) and (1.2) the equality holds if and only if \( p(z) \) has all its zeros at the origin.

If \( p(z) \) does not vanish in \( |z| < 1 \), then (1.1) and (1.2) can be respectively replaced by

\[
\begin{align*}
\max_{|z|=1} |p'(z)| & \leq \frac{n}{2} \max_{|z|=1} |p(z)|, \quad (1.3) \\
\max_{|z|=R>1} |p(z)| & \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|. \quad (1.4)
\end{align*}
\]

Inequality (1.3) was conjectured by Erdős and later proved by Lax [5], whereas inequality (1.4) is due to Ankeny and Rivlin [1]. Here in both inequalities (1.3) and (1.4), the equality holds for \( p(z) = \alpha + \beta z^n \), \( |\alpha| = |\beta| \). Inequalities (1.3) and (1.4) are, respectively, much better than inequalities (1.1) and (1.2). As a generalization of (1.3), it was shown by Malik [6] that if \( p(z) \) does not vanish in \( |z| < k, k \geq 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k} \max_{|z|=1} |p(z)|. \quad (1.5)
\]

The result is sharp and the extremal polynomial is \( p(z) = (z + k)^n \).

Chan and Malik [3] considered the class of polynomials \( p(z) = a_0 + \sum_{j=t}^{n} a_j z^j \), \( 1 \leq t \leq n \), and proved the following extension of inequality (1.5).
**Theorem 1.1.** If \( p(z) = a_0 + \sum_{j=1}^{n} a_j z^j \) is a polynomial of degree \( n \), having no zeros in the disk \( |z| < k \) where \( k \geq 1 \), then for \( 1 \leq t \leq n \),

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^t} \max_{|z|=1} |p(z)|. \tag{1.6}
\]

The result is best possible and equality holds for the polynomial \( p(z) = (z^t + k^t)^{n/t} \), where \( n \) is a multiple of \( t \).

Inequality (1.6) was also independently proved by Qazi [8, Lemma 1] who, in fact, has also proved the following result.

**Theorem 1.2.** If \( p(z) = a_0 + \sum_{j=1}^{n} a_j z^j \) is a polynomial of degree \( n \), having no zeros in the disk \( |z| < k \) where \( k \geq 1 \), then for \( 1 \leq t \leq n \),

\[
\max_{|z|=1} |p'(z)| \leq n \frac{1 + (t/n) |a_t/a_0| k^{t+1}}{1 + k^{t+1} + (t/n) |a_t/a_0| (k^{t+1} + k^{2t})} \max_{|z|=1} |p(z)|. \tag{1.7}
\]

In this paper, we improve inequality (1.7) for the class of polynomials \( p(z) = a_0 + \sum_{j=1}^{n} a_j z^j \), not vanishing in the disk \( |z| < k \), \( k \geq 1 \). More precisely, we prove the following result.

**Theorem 1.3.** If \( p(z) = a_0 + \sum_{j=1}^{n} a_j z^j \) is a polynomial of degree \( n \) which does not vanish in \( |z| < k \) where \( k \geq 1 \), then for every \( R > 1 \) and \( |z| = 1 \),

\[
|p(Rz) - p(z)| \leq (R^n - 1) \left( 1 + \frac{A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \max_{|z|=1} |p(z)| \right. \\
- \left. \left\{ 1 - \frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \right\} \frac{R^n - 1}{k^n} m \right), \tag{1.8}
\]

where \( m = \min_{|z|=1} |p(z)| \), \( 1 \leq t < n \), \( A_t = (R^t - 1)/(R^n - 1) \) and \( B_t = |a_t/a_0| \).

**Remark 1.4.** If we divide the two sides of (1.8) by \( (R - 1) \) and let \( R \to 1 \), we get

\[
\max_{|z|=1} |p'(z)| \leq n \frac{1 + (t/n) |a_t/a_0| k^{t+1}}{1 + k^{t+1} + (t/n) |a_t/a_0| (k^{t+1} + k^{2t})} \max_{|z|=1} |p(z)| \\
- \left\{ 1 - \frac{1 + (t/n) |a_t/a_0| k^{t+1}}{1 + k^{t+1} + (t/n) |a_t/a_0| (k^{t+1} + k^{2t})} \right\} \frac{mn}{k^n}, \tag{1.9}
\]

which is an improvement of (1.7) due to Qazi [8] for \( 1 \leq t < n \).

If we use the fact that

\[
|p(Rz) - p(z)| \geq |p(Rz)| - |p(z)| \tag{1.10}
\]

or

\[
|p(Rz)| \leq |p(Rz) - p(z)| + |p(z)|, \tag{1.11}
\]

the following corollary is an immediate consequence of the above theorem.
Corollary 1.5. If \( p(z) = a_0 + \sum_{j=1}^{n} a_j z^j \) is a polynomial of degree \( n \) which does not vanish in \( |z| < k \) where \( k \geq 1 \), then for \( R > 1 \)

\[
\max_{|z|=R} |p(z)| \leq \frac{R^n \{1 + A_t B_t k^{t+1}\} + k^{t+1} + A_t B_t k^{2t}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \max_{|z|=1} |p(z)| - \left\{ 1 - \frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \right\} \frac{(R^n - 1) m}{k^n},
\]

(1.12)

where \( m = \min_{|z|=k} |p(z)|, 1 \leq t < n, A_t = (R^t - 1)/(R^n - 1), \) and \( B_t = |a_t/a_0| \).

The inequality

\[
\frac{R^t - 1}{R^n - 1} \leq \frac{t}{n}
\]

(1.13)

holds for all \( R > 1 \) and \( 1 \leq t \leq n \). To prove this inequality, we observe that for every \( R > 1 \), it easily follows when \( t = n \). Hence to establish (1.13), it suffices to consider the case \( 1 \leq t \leq n - 1 \) and \( R > 1 \). So, we assume that \( R > 1 \) and \( 1 \leq t \leq n - 1 \), and then we have

\[
t R^n - n R^t = t R^t \left( R^{n-t} - 1 \right) - (n-t) \left( R^t - 1 \right)
\]

(1.14)

\[
= (R-1) \left\{ t R^t \left( R^{n-t-2} + \cdots + 1 \right) \right\} - (n-t) \left( R^{t-1} \cdots + R + 1 \right)
\]

\[
= (R-1)\left\{ t(n-t) R^t - (n-t) t R^{t-1} \right\}
\]

\[
= t(n-t)(R-1)^2 R^{t-1} > 0.
\]

This implies that \( t(R^n - 1) \geq n(R^t - 1) \), for all values of \( R > 1 \) and \( 1 \leq t \leq n - 1 \) which is equivalent to (1.13).

With the help of (1.13) a simple direct calculation yields

\[
\max_{|z|=R} |p(z)| \leq \frac{R^n \{1 + A_t B_t k^{t+1}\} + k^{t+1} + A_t B_t k^{2t}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \max_{|z|=1} |p(z)| - \left\{ 1 - \frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \right\} \frac{(R^n - 1) m}{k^n}
\]

(1.15)

Hence from Theorem 1.3, we easily deduce the following corollary.

Corollary 1.6. If \( p(z) = a_0 + \sum_{j=1}^{n} a_j z^j \) is a polynomial of degree \( n \) which does not vanish in \( |z| < k \) where \( k \geq 1 \), then for every \( R > 1 \),

\[
\max_{|z|=R} |p(z)| \leq \frac{R^n \{1 + (t/n) B_t k^{t+1}\} + k^{t+1} + (t/n) B_t k^{2t}}{1 + k^{t+1} + (t/n) B_t (k^{t+1} + k^{2t})} \max_{|z|=1} |p(z)| - \left\{ 1 - \frac{1 + (t/n) B_t k^{t+1}}{1 + k^{t+1} + (t/n) B_t (k^{t+1} + k^{2t})} \right\} \frac{(R^n - 1) m}{k^n},
\]

(1.16)

where \( m = \min_{|z|=k} |p(z)|, 1 \leq t < n, \) and \( B_t = |a_t/a_0| \).
Next, if we take \( t = 1 \) in Theorem 1.3, we get the following corollary.

**Corollary 1.7.** Let \( p(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) which does not vanish in the disk \( |z| < k \), \( k \geq 1 \), then for every \( R > 1 \)

\[
|p(Rz) - p(z)| \leq \frac{R^n - 1}{1 + k^2 + A_1 B_1 (2k^2)} \max_{|z|=1} |p(z)|
- \left( 1 - \frac{1 + A_1 B_1 k^2}{1 + k^2 + A_1 B_1 (2k^2)} \right) (R^n - 1) m k^n,
\]

where \( m = \min_{|z|=k} |p(z)| \), \( A_1 = (R - 1)/(R^n - 1) \), and \( B_1 = |a_1/a_0| \).

**Remark 1.8.** If we divide the two sides of (1.17) by \( (R - 1) \) and let \( R \to 1 \), it easily follows that, if \( p(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n \) such that \( p(z) \neq 0 \) in \( |z| < k \), \( k \geq 1 \), then for \( |z| \leq 1 \),

\[
|p'(z)| \leq n \frac{n |a_0| + k^2 |a_1|}{n(1 + k^2) |a_0| + 2k^2 |a_1|} \max_{|z|=1} |p(z)|
- \left( 1 - \frac{n |a_0| + k^2 |a_1|}{n(1 + k^2) |a_0| + 2k^2 |a_1|} \right) mn k^n
\]

which is an improvement of a result due to Govil et al. [4].

It is known that

\[
\frac{t}{n} \left| \frac{a_t}{a_0} \right| k^t \leq 1.
\]

(1.19)

Using this fact and inequality (1.13), it is easy to verify that

\[
\frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \leq \frac{1}{1 + k^t}.
\]

(1.20)

By using these observations, the following result is an immediate consequence of Theorem 1.3.

**Corollary 1.9.** If \( p(z) = a_0 + \sum_{j=t}^{n} a_j z^j \) is a polynomial of degree \( n \) which does not vanish in the disk \( |z| < k \) where \( k \geq 1 \), then for every \( R > 1 \) and \( |z| = 1 \),

\[
|p(Rz) - p(z)| \leq \frac{R^n - 1}{1 + k^t} \max_{|z|=1} |p(z)| - \left( 1 - \frac{1}{1 + k^t} \right) (R^n - 1) m k^n
\]

\[
= \frac{R^n - 1}{1 + k^t} \left\{ \max_{|z|=1} |p(z)| - \frac{m}{k^{n-t}} \right\}
\]

and in the fortiori

\[
\max_{|z|=R} |p(z)| \leq \frac{R^n + k^t}{1 + k^t} \max_{|z|=1} |p(z)| - \left( \frac{R^n - 1}{1 + k^t} \right) \frac{m}{k^{n-t}}.
\]

(1.22)
**Remark 1.10.** For \( k = t = 1 \), (1.22) reduces to

\[
M(p, R) \leq \frac{R^n + 1}{2} M(p, 1) - \left( \frac{R^n - 1}{2} \right) m,
\]

which is an improvement of (1.4) due to Ankeny and Rivlin [1].

Inequality (1.23) was proved by Aziz and Dawood [2].

2. A lemma

**Lemma 2.1.** Let \( p(z) = a_0 + \sum_{j=1}^{n} a_jz^j \) be a polynomial of degree \( n \) which does not vanish in \( |z| < l \) where \( k \geq 1 \), then for every \( R > 1 \) and \( |z| = 1 \),

\[
|p(Rz) - p(z)| \leq (R^n - 1) \frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \max_{|z|=1} |p(z)|,
\]

where \( 1 \leq t \leq n \), \( A_t = (R^t - 1) / (R^n - 1) \) and \( B_t = |a_t / a_0| \).

This lemma is due to Shah [10].

3. Proof of Theorem 1.3.** By Rouche’s theorem, the polynomial \( p(z) + m\beta z^n \), \( |\beta| < 1/k^n \), has no zero in \( |z| < k \) for \( k \geq 1 \). So on applying Lemma 2.1 to the polynomial \( p(z) + m\beta z^n \), \( |\beta| < 1/k^n \), we get

\[
|p(Rz) - p(z) + m\beta z^n (R^n - 1)| \leq (R^n - 1) \frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \max_{|z|=1} \{ |p(z)| + |m\beta z^n| \}.
\]

Now choosing the argument of \( \beta \) suitably, the above inequality becomes

\[
|p(Rz) - p(z)| + |m\beta z^n (R^n - 1)| \leq (R^n - 1) \frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \max_{|z|=1} \{ |p(z)| + m|\beta| \}.
\]

Finally letting \( |\beta| \to 1/k^n \), we get the desired result. \( \square \)
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References


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