SUBMANIFOLDS OF $F$-STRUCTURE MANIFOLD SATISFYING

$$F^K + (-)^{K+1} F = 0$$

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Abstract. The purpose of this paper is to study invariant submanifolds of an $n$-dimensional manifold $M$ endowed with an $F$-structure satisfying $F^K + (-)^{K+1} F = 0$ and $F^W + (-)^{W+1} F \neq 0$ for $1 < W < K$, where $K$ is a fixed positive integer greater than 2. The case when $K$ is odd ($\geq 3$) has been considered in this paper. We show that an invariant submanifold $\tilde{M}$, embedded in an $F$-structure manifold $M$ in such a way that the complementary distribution $D_m$ is never tangential to the invariant submanifold $\Psi(\tilde{M})$, is an almost complex manifold with the induced $\tilde{F}$-structure. Some theorems regarding the integrability conditions of induced $\tilde{F}$-structure are proved.

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1. Introduction. Invariant submanifolds have been studied by Blair et al. [1], Kubo [4], Yano and Okumura [7, 8], and among others. Yano and Ishihara [6] have studied and shown that any invariant submanifold of codimension 2 in a contact Riemannian manifold is also a contact Riemannian manifold. We consider an $F$-structure manifold $M$ and study its invariant submanifolds. Let $F$ be a nonzero tensor field of the type $(1, 1)$ and of class $C^\infty$ on an $n$-dimensional manifold $M$ such that (see [3])

$$F^K + (-)^{K+1} F = 0, \quad F^W + (-)^{W+1} F \neq 0, \quad \text{for } 1 < W < K, \quad (1.1)$$

where $K$ is a fixed positive integer greater than 2. Such a structure on $M$ is called an $F$-structure of rank $r$ and of degree $K$. If the rank of $F$ is constant and $r = r(F)$, then $M$ is called an $F$-structure manifold of degree $K(\geq 3)$.

Let the operator on $M$ be defined as follows (see [3])

$$\ell = (-)^{K} F^{K-1}, \quad m = I + (-)^{K+1} F^{K-1}, \quad (1.2)$$

where $I$ denotes the identity operator on $M$. For the operators defined by (1.2), we have

$$\ell + m = I, \quad \ell^2 = \ell; \quad m^2 = m. \quad (1.3)$$

For $F$ satisfying (1.1), there exist complementary distribution $D_\ell$ and $D_m$ corresponding to the projection operators $\ell$ and $m$, respectively. If rank$(F) = \text{constant}$ on $M$, then $\dim D_\ell = r$ and $\dim D_m = (n - r)$. We have the following results (see [3]).

$$F\ell = \ell F = F, \quad Fm = m F = 0, \quad (1.4a)$$

$$F^{K-1} = (-)^K \ell, \quad F^{K-1} \ell = -\ell; \quad F^{K-1} m = 0. \quad (1.4b)$$

Thus $F^{K-1}$ acts on $D_\ell$ as an almost complex structure and on $D_m$ as a null operator.
2. Invariant submanifolds of $F$-structure manifold. Let $\tilde{M}$ be a differentiable manifold embedded differentially as a submanifold in an $n$-dimensional $C^\infty$ Riemannian manifold $M$ with an $F$-structure and we denote its embedding by $\Psi: \tilde{M} \to M$. Denote by $B: T(\tilde{M}) \to T(M)$ the differential mapping of $\Psi$, where $d\Psi = B$ is the Jacobson map of $\Psi$. $T(\tilde{M})$ and $T(M)$ are tangent bundles of $\tilde{M}$ and $M$, respectively. We call $T(\tilde{M},M)$ as the set of all vectors tangent to the submanifold $\Psi(\tilde{M})$. It is known that $B: T(\tilde{M}) \to T(\tilde{M},M)$ is an isomorphism (see [5]).

Let $\tilde{X}$ and $\tilde{Y}$ be two $C^\infty$ vector fields defined along $\Psi(\tilde{M})$ and tangent to $\Psi(\tilde{M})$. Let $X$ and $Y$ be the local extensions of $\tilde{X}$ and $\tilde{Y}$. The restriction of $[X,Y]_{\tilde{M}}$ is determined independently of the choice of these local extensions $X$ and $Y$. Therefore, we can define

$$[\tilde{X}, \tilde{Y}] = [X,Y]_{\tilde{M}}.$$ (2.1)

Since $B$ is an isomorphism, it is easy to see that $[B\tilde{X}, B\tilde{Y}] = B[X, Y]$ for all $\tilde{X}, \tilde{Y} \in T(\tilde{M})$. We denote by $G$ the Riemannian metric tensor of $M$ and put

$$\tilde{g}(\tilde{X}, \tilde{Y}) = g(B\tilde{X}, B\tilde{Y}) \quad \forall \tilde{X}, \tilde{Y} \in T(\tilde{M}),$$ (2.2)

where $g$ is the Riemannian metric in $M$ and $\tilde{g}$ is the induced metric of $\tilde{M}$.

**Definition 2.1.** We say that $\tilde{M}$ is an invariant submanifold of $M$ if

(i) the tangent space $T_p(\Psi(\tilde{M}))$ of the submanifold $\Psi(\tilde{M})$ is invariant by the linear mapping $F$ at each point $p$ of $\Psi(\tilde{M})$,

(ii) for each $\tilde{X} \in T(\tilde{M})$, we have

$$F^{(K-1)/2}(B\tilde{X}) = B\tilde{X}^t.$$ (2.3)

**Definition 2.2.** Let $\tilde{F}$ be a $(1,1)$-tensor field defined in $\tilde{M}$ such that $\tilde{F}(\tilde{X}) = \tilde{X}^t$ and $M$ is an invariant submanifold, then we have

$$F(B\tilde{X}) = B(\tilde{F}\tilde{X}),$$ (2.4a)

$$F^{(K-1)/2}(B\tilde{X}) = B(\tilde{F}^{(K-1)/2}\tilde{X}).$$ (2.4b)

We see that there are two cases for any invariant submanifold $\tilde{M}$. We assume the following cases.

**Case 1.** The distribution $D_m$ is never tangential to $\Psi(\tilde{M})$.

**Case 2.** The distribution $D_m$ is always tangential to $\Psi(\tilde{M})$.

We will consider Case 1 and assume that no vector field of the type $mX$, where $X \in T(\Psi(\tilde{M}))$ is tangential to $\Psi(\tilde{M})$.

**Theorem 2.3.** An invariant submanifold $\tilde{M}$ is an almost complex manifold if the following two conditions are satisfied:

(i) the distribution $D_m$ is never tangential to $\Psi(\tilde{M})$, and

(ii) $\tilde{F}$ in $\tilde{M}$ defines an induced almost complex structure satisfying $\tilde{F}^{K-1} = (-)^K I$.

**Proof.** Applying $F^{(K-1)/2}$ in (2.4), we obtain

$$F^{(K-1)/2}(F^{(K-1)/2}(B\tilde{X})) = F^{(K-1)/2}(B(\tilde{F}^{(K-1)/2}\tilde{X})).$$ (2.5)
Making use of (2.4a) in (2.5), we get
\[ F^{K-1}(B\tilde{X}) = B(\tilde{F}^{K-1}\tilde{X}). \] (2.6)

In order to show that vector fields of the type \( B\tilde{X} \) belong to the distribution \( D_\ell \), we suppose that \( m(B\tilde{X}) \neq 0 \), then using (1.2) we have
\[ m(B\tilde{X}) = (I + (-)^{K+1}F^{K-1})B\tilde{X} = B\tilde{X} + (-)^{K+1}F^{K-1}(B\tilde{X}) \] (2.7)
which in view of (2.6) becomes
\[ m(B\tilde{X}) = B\tilde{X} + (-)^{K+1}B(\tilde{F}^{K-1}\tilde{X}) = B[\tilde{X} + (-)^{K+1}\tilde{F}^{K-1}\tilde{X}] \] (2.8)
which, contrary to our assumption, shows that \( m(B\tilde{X}) \) is tangential to \( \Psi(\tilde{M}) \). Thus \( m(B\tilde{X}) = 0 \).

Also, in view of (1.4b), (1.3), and (2.6) we obtain
\[ B(\tilde{F}^{K-1}\tilde{X}) = F^{K-1}(B\tilde{X}) = (-)^{K}\ell(B\tilde{X}) = (-)^{K}(I - m)B\tilde{X} \]
\[ = (-)^{K}B\tilde{X} - (-)^{K}mB\tilde{X}, \]
\[ B(\tilde{F}^{K-1}\tilde{X}) = (-)^{K}B\tilde{X}. \] (2.9)

Since \( B \) is an isomorphism, we get
\[ \tilde{F}^{K-1} = (-)^{K}I. \] (2.10)

Let \( \mathcal{F}(M) \) be the ring of real-valued differentiable functions on \( M \), and let \( \mathcal{X}(M) \) be the module of derivatives of \( \mathcal{F}(M) \). Then \( \mathcal{X}(M) \) is Lie algebra over the real numbers and the elements of \( \mathcal{X}(M) \) are called vector fields. Then \( M \) is equipped with \((1,1)\)-tensor field \( F \) which is a linear map such that
\[ F : \mathcal{X}(M) \longrightarrow \mathcal{X}(M). \] (2.11)

Let \( M \) be of degree \( K \) and let \( K \) be a positive odd integer greater than 2. Then we consider a positive definite Riemannian metric with respect to which \( D_\ell \) and \( D_m \) are orthogonal so that
\[ g(X,Y) = g(HX,HY) + g(mX,Y), \] (2.12)
where \( H = F^{(K-1)/2} \) for all \( X,Y \in \mathcal{X}(M) \). \( \square \)

**Definition 2.4.** The induced metric \( \tilde{g} \) defined by (2.2) is Hermitian if the following is satisfied:
\[ \tilde{g}(H\tilde{X},H\tilde{Y}) = \tilde{g}(\tilde{X},\tilde{Y}), \quad \text{where} \quad H = F^{(K-1)/2}. \] (2.13)

**Theorem 2.5.** If \( F \)-structure manifold has the following two properties, that is,
(1) \( \tilde{M} \) is an invariant submanifold of \( F \)-structure manifold \( M \) such that distribution \( D_m \) is never tangential to \( \Psi(\tilde{M}) \),
(2) the Riemannian metric \( g \) on \( M \) is defined by (2.12).
Then the induced metric \( \tilde{g} \) of \( \tilde{M} \) defined by (2.2) is Hermitian.
In view of (2.2) and (2.13) we obtain
\[ \tilde{g}(F^{(K-1)/2}\bar{X},F^{(K-1)/2}\bar{Y}) = g(B\tilde{F}^{(K-1)/2}\bar{X},B\tilde{F}^{(K-1)/2}\bar{Y}). \] (2.14)

Applying (2.4) and (2.12) in (2.14), we get
\[ \tilde{g}(F^{(K-1)/2}\bar{X},F^{(K-1)/2}\bar{Y}) = g(F^{(K-1)/2}B\bar{X},F^{(K-1)/2}B\bar{Y}) \]
\[ = g(B\bar{X},B\bar{Y}) - g(mB\bar{X},B\bar{Y}). \] (2.15)

Since the distribution \( D_m \) is never tangential to \( \Psi(\bar{M}) \), on using (2.2) we get
\[ \tilde{g}(F^{(K-1)/2}\bar{X},F^{(K-1)/2}\bar{Y}) = g(B\tilde{X},B\tilde{Y}) = \tilde{g}(\tilde{X},\tilde{Y}). \] (2.16)

Now, we consider the second case and assume that the distribution \( D_m \) is always tangential to \( \Psi(\bar{M}) \). In view of Case 2, we have \( m(B\tilde{X}) = B\tilde{X}^* \), where \( \tilde{X}^* \in T(\bar{M}) \) for some \( \tilde{X}^* \in T(\bar{M}) \).

We define (1,1)-tensor fields \( \tilde{m} \) and \( \tilde{\ell} \) in \( \bar{M} \) as follows:
\[ \tilde{\ell} = (-)^K\tilde{F}^{K-1}, \quad \tilde{m} = \tilde{I} + (-)^{K+1}\tilde{F}^{K-1}, \] (2.17a)
\[ \tilde{m}\tilde{X} = \tilde{X}^*, \quad m(B\tilde{X}) = B(\tilde{m}\tilde{X}). \] (2.17b)

**Theorem 2.6.** We have
\[ B(\tilde{\ell}\tilde{X}) = \ell(B\tilde{X}). \] (2.18)

**Proof.** In view of (2.17a), equation (2.18) assumes the following form:
\[ B(\tilde{\ell}\tilde{X}) = B((-)^K\tilde{F}^{K-1}\tilde{X}) = (-)^KB(\tilde{F}^{K-1}\tilde{X}). \] (2.19)

Making use of (2.6) and (2.15) in (2.19), we get
\[ B(\tilde{\ell}\tilde{X}) = (-)^K\tilde{F}^{K-1}(B\tilde{X}) = \tilde{\ell}(B\tilde{X}). \] (2.20)

**Theorem 2.7.** For \( \tilde{\ell} \) and \( \tilde{m} \) satisfying (2.17a), we have
\[ \tilde{\ell} + \tilde{m} = \tilde{I}, \quad \tilde{\ell}^2 = \tilde{\ell}, \quad \tilde{m}^2 = \tilde{m}. \] (2.21)

**Proof.** From (1.3) we have \( \ell + m = I \), which can be written as \( (\ell + m)B\tilde{X} = B\tilde{X} \), thus we have
\[ \ell B\tilde{X} + mB\tilde{X} = B\tilde{X} \] (2.22)
which in view of (2.17b) and (2.18) becomes
\[ B(\tilde{\ell}\tilde{X}) + B(\tilde{m}\tilde{X}) = B(\tilde{\ell} + \tilde{m})\tilde{X} = B\tilde{X}. \] (2.23)

Therefore \( \tilde{\ell} + \tilde{m} = \tilde{I} \) since \( B \) is an isomorphism. Proof of the other relations follows in a similar manner.

**Theorem 2.7** shows that \( \tilde{\ell} \) and \( \tilde{m} \) defined by (2.17a) are complementary projection operators on \( \bar{M} \).
**Theorem 2.8.** If $F$-structure manifold has the following property, that is, $\tilde{M}$ is an invariant submanifold of $F$-structure manifold $M$ such that distribution $D_m$ is always tangential to $\Psi(\tilde{M})$. Then there exists an induced $\tilde{F}$-structure manifold which admits a similar Riemannian metric $\tilde{g}$ satisfying

$$\tilde{g}(\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{H}\tilde{X}, \tilde{H}\tilde{Y}) + \tilde{g}(m\tilde{X}\tilde{Y}).$$

(2.24)

**Proof.** From (2.4b) we get

$$B(\tilde{F}^{(K-1)/2} \tilde{X}) = F^{(K-1)/2}(B\tilde{X}).$$

(2.25)

Furthermore,

$$B(\tilde{F}^K \tilde{X}) = F^K(B\tilde{X})$$

(2.26)

which in view of (1.1) and (2.4a) yields

$$B(\tilde{F}^K \tilde{X}) = B(-(-)^{K+1}F\tilde{X})$$

(2.27)

which shows that $\tilde{F}$ defines an $\tilde{F}$-structure manifold which satisfies

$$\tilde{F}^K + (-)^{K+1}\tilde{F} = 0.$$  

(2.28)

In consequence of (2.2), (2.4b), and (2.12) we obtain

$$\tilde{g}(\tilde{H}, \tilde{X}, \tilde{H}\tilde{Y}) + \tilde{g}(m\tilde{X}, \tilde{Y}) = g(BH\tilde{X}, BH\tilde{Y}) + g(Bm\tilde{X}, B\tilde{Y})$$

$$= g(HB\tilde{X}, HB\tilde{Y}) + g(mB\tilde{X}, B\tilde{Y})$$

$$= g(B\tilde{X}, B\tilde{Y}), \text{ where } \tilde{H} = \tilde{F}^{(K-1)/2}$$

(2.29)

which in view of the fact that $B$ is an isomorphism gives

$$\tilde{g}(\tilde{H}, \tilde{X}, \tilde{H}\tilde{Y}) + \tilde{g}(m\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}).$$

(2.30)


3. **Integrability conditions.** The Nijenhuis tensor $N$ of the type (1.2) of $F$ satisfying (1.1) in $M$ is given by (see [2])

$$N(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y],$$

(3.1)

and the Nijenhuis tensor $\tilde{N}$ of $\tilde{F}$ satisfying (2.28) in $\tilde{M}$ is given by

$$N(\tilde{X}, \tilde{Y}) = [\tilde{F}\tilde{X}, \tilde{F}\tilde{Y}] - \tilde{F}[\tilde{F}\tilde{X}, \tilde{Y}] - \tilde{F}[\tilde{X}\tilde{F}\tilde{Y}] + \tilde{F}^2[\tilde{X}, \tilde{Y}].$$

(3.2)

**Theorem 3.1.** The Nijenhuis tensors $N$ and $\tilde{N}$ of $M$ and $\tilde{M}$ given by (3.1) and (3.2) satisfy the following relation:

$$N(B\tilde{X}, B\tilde{Y}) = B\tilde{N}(\tilde{X}, \tilde{Y}).$$

(3.3)

**Proof.** We have

$$N(B\tilde{X}, B\tilde{Y}) = [F(B\tilde{X}), F(B\tilde{Y})] - F[F(B\tilde{X}), B\tilde{Y}] - F[B\tilde{X}, F(B\tilde{Y})] + F^2[B\tilde{X}, B\tilde{Y}]$$

(3.4)
which in view of (2.4a) becomes

\[ N(B\hat{X},B\hat{Y}) = B[\hat{\mathcal{F}}\hat{X},\hat{\mathcal{F}}\hat{Y}] - F[B(\hat{\mathcal{F}}\hat{X}),B\hat{Y}] - F[(B\hat{X},B\hat{F}\hat{Y})] + F^2[B\hat{X},B\hat{Y}] \]

\[ = B[\hat{\mathcal{F}}\hat{X},\hat{\mathcal{F}}\hat{Y}] - FB[\hat{\mathcal{F}}\hat{X},\hat{\mathcal{F}}\hat{Y}] - FB[\hat{\mathcal{F}}\hat{X},\hat{\mathcal{F}}\hat{Y}] + BF^2[\hat{\mathcal{F}}\hat{X},\hat{\mathcal{F}}\hat{Y}] \]

\[ = B[\hat{\mathcal{F}}\hat{X},\hat{\mathcal{F}}\hat{Y}] - B\hat{\mathcal{F}}[\hat{\mathcal{F}}\hat{X},\hat{\mathcal{F}}\hat{Y}] + B\hat{\mathcal{F}}^2[\hat{\mathcal{F}}\hat{X},\hat{\mathcal{F}}\hat{Y}] = B\tilde{N}(\hat{\mathcal{F}}\hat{X},\hat{\mathcal{F}}\hat{Y}). \]  

**Theorem 3.2.** The following identities hold:

\[ B\tilde{N}(\hat{\mathcal{F}}\hat{X},\hat{\mathcal{F}}\hat{Y}) = N(\hat{\mathcal{F}}B\hat{X},\hat{\mathcal{F}}B\hat{Y}), \quad B\tilde{N}(\hat{m}\hat{X},\hat{m}\hat{Y}) = N(mB\hat{X},mB\hat{Y}), \quad B\{m\tilde{N}(\hat{\mathcal{F}}\hat{X},\hat{\mathcal{F}}\hat{Y})\} = mN(B\hat{X},B\hat{Y}). \]  

**Proof.** The proof of (3.6) follows by virtue of Theorem 3.1, equations (1.4a), (2.4a), (2.17a), (2.17b), and (3.3). 

For \( \hat{\mathcal{F}} \) satisfying (2.28), there exists complementary distribution \( D_\ell \) and \( D_m \) corresponding to the projection operators \( \ell \) and \( m \) in \( \hat{\mathcal{M}} \) given by (2.17a). Then in view of the integrability conditions of \( \hat{\mathcal{F}} \) structure we state the following theorems.

**Theorem 3.3.** If \( D_\ell \) is integrable in \( M \), then \( D_\ell \) is also integrable in \( \hat{\mathcal{M}} \). If \( D_m \) is integrable in \( M \), then \( D_m \) is also integrable in \( \hat{\mathcal{M}} \).

**Theorem 3.4.** If \( D_\ell \) and \( D_m \) are both integrable in \( M \), then \( D_\ell \) and \( D_m \) are also integrable in \( \hat{\mathcal{M}} \).

**Theorem 3.5.** If \( F \)-structure is integrable in \( M \), then the induced structure \( \hat{F} \) is also integrable in \( \hat{\mathcal{M}} \).

**References**


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