PAIRS OF PATHS AND CRITICAL POINTS

FLORIN CARAGIU and IOANA CARAGIU

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ABSTRACT. Two sufficient conditions are presented, in terms of the values taken by a holomorphic function \( f(z) \) on a pair of smooth paths intersecting at a point \( z_0 \) in its domain, implying that \( f'(z_0) = 0 \).

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In the present paper, we present two sufficient conditions expressed in terms of the values taken by a holomorphic function \( f \) on a pair of smooth paths intersecting at a point \( z_0 \) in the domain of \( f \), with tangent vectors at \( z_0 \) linearly independent over \( \mathbb{R} \), implying that \( f'(z_0) = 0 \).

**Theorem 1.** Let \( f : D \subset \mathbb{C} \to \mathbb{C} \) be a holomorphic function, where \( D \subset \mathbb{C} \) is a domain and let \( \gamma, \Gamma : (0, 1) \to D \) be two smooth (\( C^1 \)) paths. Assume the following:

(i) for a certain \( z_0 \in D \) and some \( t_1, t_2 \in (0, 1) \) we have \( z_0 = \gamma(t_1) = \Gamma(t_2) \);
(ii) \( \gamma'(t_1) \) and \( \Gamma'(t_2) \) linearly independent over \( \mathbb{R} \) (i.e., non-collinear),
(iii) \( |f(z)| \) takes a constant value on the subset \( \gamma((0, 1)) \cup \Gamma((0, 1)) \) of \( D \). Then \( f'(z_0) = 0 \).

**Proof.** Let \( f = u + iv, \gamma = \gamma_1 + i\gamma_2, \) and \( \Gamma = \Gamma_1 + i\Gamma_2, \) where \( u, v \) are real-valued functions while \( \gamma_1, \gamma_2, \Gamma_1, \Gamma_2 \) are real-valued smooth paths. The assumption (iii) can be written as

\[
u^2(\gamma(t)) + v^2(\gamma(t)) = u^2(\Gamma(t)) + v^2(\Gamma(t)) = c \tag{1}\]

for any \( t \in (0, 1) \), where \( c \) is some constant. Note first that if \( c = 0 \), from (1) together with the identity theorem of the holomorphic functions it follows that \( f(z) = 0 \) for any \( z \in D \). This being the case, we assume \( c \neq 0 \) from now on. We differentiate (1) with respect to \( t \). We then have, for any \( t \in (0, 1) \),

\[
\frac{d}{dt}(u^2(\gamma(t)) + v^2(\gamma(t))) = 0, \tag{2}
\]

that is, by using the chain rule,

\[
2u(\gamma(t))u_x(\gamma(t))\gamma'_1(t) + 2u(\gamma(t))u_y(\gamma(t))\gamma'_2(t)
+ 2v(\gamma(t))v_x(\gamma(t))\gamma'_1(t) + 2v(\gamma(t))v_y(\gamma(t))\gamma'_2(t) = 0 \tag{3}
\]

together with the similar relation for \( \Gamma \):

\[
2u(\Gamma(t))u_x(\Gamma(t))\Gamma'_1(t) + 2u(\Gamma(t))u_y(\Gamma(t))\Gamma'_2(t)
+ 2v(\Gamma(t))v_x(\Gamma(t))\Gamma'_1(t) + 2v(\Gamma(t))v_y(\Gamma(t))\Gamma'_2(t) = 0 \tag{4}
\]
holding also for any \( t \in (0, 1) \). By using the Cauchy-Riemann equations in (3) and (4), respectively, we get, after a convenient grouping of terms,

\[
\begin{align*}
\mathbf{u}(\nu(t)) [u_x(\nu(t)) y'_2(t) - u_x(\nu(t)) y'_1(t)] + v(\nu(t)) [u_x(\nu(t)) y'_2(t) + v_x(\nu(t)) y'_1(t)] &= 0, \\
u(\Gamma(t)) [u_x(\Gamma(t)) \Gamma'_2(t) - u_x(\Gamma(t)) \Gamma'_1(t)] + v(\Gamma(t)) [u_x(\Gamma(t)) \Gamma'_2(t) + v_x(\Gamma(t)) \Gamma'_1(t)] &= 0,
\end{align*}
\]

for any \( t \in (0, 1) \). By specializing \( t = t_1 \) in (5) and \( t = t_2 \) in (6), we obtain

\[
\begin{align*}
\mathbf{u}(z_0) [u_x(z_0) y'_1(t_1) - u_x(z_0) y'_2(t_1)] + v(z_0) [u_x(z_0) y'_2(t_1) + v_x(z_0) y'_1(t_1)] &= 0, \\
u(z_0) [u_x(z_0) \Gamma'_1(t_2) - u_x(z_0) \Gamma'_2(t_2)] + v(z_0) [u_x(z_0) \Gamma'_2(t_2) + v_x(z_0) y'_1(t_2)] &= 0.
\end{align*}
\]

Since \( u^2(z_0) + v^2(z_0) = c \neq 0 \), it follows from (7) that

\[
\begin{cases}
\mathbf{u}(z_0), v(z_0) = (0, 0) \\
\mathbf{u}(z_0), v(z_0) = (0, 0) \quad \text{is a nontrivial solution of the linear homogeneous system}
\end{cases}
\]

\[
\begin{align*}
X[u_x(z_0) y'_1(t_1) - u_x(z_0) y'_2(t_1)] + Y[u_x(z_0) y'_2(t_1) + v_x(z_0) y'_1(t_1)] &= 0, \\
X[u_x(z_0) \Gamma'_2(t_2)] + Y[u_x(z_0) \Gamma'_2(t_2) + v_x(z_0) y'_1(t_2)] &= 0,
\end{align*}
\]

and so

\[
\begin{vmatrix}
u(z_0) & u_x(z_0) & u_x(z_0) y'_2(t_1) + v_x(z_0) y'_1(t_1) \\
u(z_0) & u_x(z_0) & u_x(z_0) \Gamma'_2(t_2) + v_x(z_0) y'_1(t_2)
\end{vmatrix} = 0.
\]

By expanding the determinant, equation (10) can be rewritten as

\[
(u^2(z_0) + v^2(z_0))(y'_1(t_1) \Gamma'_2(t_2) - \Gamma'_1(t_2) y'_2(t_1)) = 0.
\]

On the other hand, the assumption (iii) can be rewritten as

\[
\begin{vmatrix}
y'_1(t_1) & y'_2(t_1) \\
\Gamma'_2(t_2) & \Gamma'_1(t_2)
\end{vmatrix} = 0.
\]

Finally, from (11) and (12) it follows that

\[
u^2(z_0) + v^2(z_0) = 0,
\]

that is, \( u_x(z_0) = v_x(z_0) = 0 \). This, together with the Cauchy-Riemann relations [1] implies \( u_y(z_0) = v_x(z_0) = 0 \) and so \( f'(z_0) = 0 \). This concludes the proof of Theorem 1.

The following exercise represents an interesting corollary of Theorem 1.

**Corollary 2.** Let \( D \subset \mathbb{C} \) be a domain which contains the square \([-1, 1] \times [-1, 1]\). Assume that \( f : D \to \mathbb{C} \) is a holomorphic function with the property that there exists \( c \in \mathbb{R}^+ \) such that

\[
|f(x + i0)| = c = |f(x + i \sin(\frac{1}{x}))|
\]

for any \( x \in (0, 1) \). Then \( f \) is a constant function.
**Proof.** Let \( y, \Gamma : (0,1) \to \mathbb{C} \) defined by

\[
y'(t) = (t,0), \quad \Gamma(t) = \left( t, \sin \left( \frac{1}{t} \right), \right),
\]

respectively. We have

\[
y'(t) = (1,0), \quad \Gamma'(t) = \left( 1, -\frac{1}{t^2} \cos \left( \frac{1}{t} \right) \right),
\]

for any \( t \in (0,1) \). Consider the sequence

\[
t_k = \frac{1}{k\pi} \in (0,1)
\]

convergent to 0. This choice of the sequence makes sure that

\[
y(t_k) = \Gamma(t_k) = (t_k,0)
\]

for any \( k \geq 1 \). We also have \( y'(t_k) = (1,0) \) and \( \Gamma'(t_k) = (1, -k^2(-1)^k \pi^2) \) which implies immediately that \( y(t_k) \) and \( \Gamma(t_k) \) are linearly independent over \( \mathbb{R} \) for any \( k \geq 1 \). By Theorem 1,

\[
f'(t_k + i0) = 0
\]

holds true for any \( k \geq 1 \). Since \( f' \) is holomorphic and \( t_k \to 0 \in D \) is an accumulation point for the zeros of \( f' \), it follows that \( f'(z) = 0 \) for any \( z \in D \), that is, \( f \) is a constant on \( D \).

Another result of similar flavour is the following theorem.

**Theorem 3.** Let \( f : \mathbb{C} \to \mathbb{C} \) be holomorphic on an open neighborhood \( V \) of \( z_0 \), and let \( y_1, y_2 : (0,1) \to V \) be a pair of \( C^1 \) paths such that for some \( t_1, t_2 \in (0,1) \), we have \( y_1(t_1) = y_2(t_2) = z_0 \) and \( y_1'(t_1), y_2'(t_2) \) are linearly independent over \( \mathbb{R} \). We also assume that \( f(y_k(t)) \in \mathbb{R}, k = 1,2 \) for any \( t \in (0,1) \). Then, under the above assumptions, \( f'(z_0) = 0 \). If, in addition, \( \arg(y_1'), \arg(y_2') \) are constant functions, then there exists a nonnegative integer \( n \) and a holomorphic function \( h \) defined on some open neighborhood of 0 such that \( f(z) = h((z - z_0)^n) \) for \( z \in V \).

**Proof.** Let \( \phi \) be the angle between \( y_1'(t_1) \) and \( y_2'(t_2) \). Consider two sequences \( \{x_n\}, \{y_n\} \) of numbers from \((0,1)\) such that \( \lim_{n\to\infty} x_n = t_1 \) while \( \lim_{n\to\infty} y_n = t_2 \). Then

\[
f'(z_0) = \lim_{n\to\infty} \frac{f(y_1(x_n)) - f(y_1(t_1))}{x_n - y_1(t_1)} = \lim_{n\to\infty} \frac{(f(y_1(x_n)) - f(y_1(t_1)))((x_n - t_1))}{(y_1(x_n) - y_1(t_1))(x_n - t_1)} \in \mathbb{R}e^{-i\arg(y_1'(t_1))}.
\]

In a similar way, it is shown that

\[
f'(z_0) \in \mathbb{R}e^{-i\arg(y_2'(t_2))}.
\]

From (20) and (21), together with the assumption that \( y_1'(t_1) \) and \( y_2'(t_2) \) are linearly independent over \( \mathbb{R} \), it follows that \( f'(z_0) \) has to be zero. This concludes the proof of the
first part of the theorem. We assume now that \( \arg(y_1^t), \arg(y_2^t) \) are constant functions, say \( \arg(y_k^t) = c_k, \ k = 1, 2, \) where \( c_1 \neq c_2. \) Then, keeping in mind that \( f(y_k(t)) \in \mathbb{R}, \ k = 1, 2 \) for any \( t \in (0, 1), \) we see that

\[
f^{(r)}(y_k(t)) \in \mathbb{R}e^{-ic_k}
\]

for \( k = 1, 2 \) and \( t \in (0, 1). \) By induction on \( r, \) we can show that

\[
f^{(r)}(y_k(t)) \in \mathbb{R}e^{-irc_k}
\]

holds true for any nonnegative integer \( r \) where \( k = 1, 2 \) and \( t \in (0, 1). \) Indeed, for \( r = 0 \) and \( r = 1, \) equation (23) is already shown. Assuming that (23) is true, by differentiation we get

\[
f^{(r+1)}(y_k(t))y_k'(t) \in \mathbb{R}e^{-irc_k}.
\]

From (24) and the fact that \( \arg(y_k'(t)) = c_k, \) it follows that

\[
f^{(r+1)}(y_k(t)) \in \mathbb{R}e^{-i(r+1)c_k}
\]

which concludes the inductive proof of (23). By specializing \( t = t_1 \) and then \( t = t_2 \) in (23), it follows that

\[
f^{(r)}(z_0) \in \mathbb{R}e^{-irc_1} \cap \mathbb{R}e^{-irc_2}
\]

for any \( r = 0, 1, 2, \ldots. \) From (26) it follows that, for any given \( r, \) either \( f^{(r)}(z_0) = 0 \) or \( e^{ir\phi} \in \mathbb{R} \) (i.e., \( r\phi \in 2\pi\mathbb{Z} \)). At this moment we distinguish two cases. First, if \( \phi/\pi \in \mathbb{R} \setminus \mathbb{Q}, \) it follows that \( f^{(r)}(z_0) = 0 \) for any \( r = 0, 1, 2, \ldots \) which implies that \( f(z) \) is constant on a neighborhood of \( z_0 \) and this being the case the choice \( h = \text{constant} = c \) would work. We consider now the second case, when \( \phi = m\pi/n, \) where \( 0 < m < n, \ m, n \in \mathbb{Z}_{>0}, \ (m, n) = 1. \) From (26) it follows that \( f^{(r)}(z_0) = 0 \) for any \( r \) which is not divisible by \( n, \) since in this case \( e^{ir\phi} = e^{irm\pi/n} \not\in \mathbb{R}. \) Therefore, on some neighborhood of \( z_0 \) the power series expansion of \( f \) has the form

\[
f(z) = \sum_{l=0}^{\infty} a_{ln}(z-z_0)^{ln} = \sum_{l=0}^{\infty} a_{ln}[(z-z_0)^{n}]^l.
\]

If we denote

\[
h(z) := \sum_{l=0}^{\infty} a_{ln}z^l,
\]

it follows that \( h \) is holomorphic on some neighborhood of 0 and satisfies \( f(z) = h((z-z_0)^n). \) This concludes the proof of Theorem 3.

\[
\square
\]

**References**
