ON HENSTOCK-DUNFORD AND HENSTOCK-PETTIS INTEGRALS

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ABSTRACT. We give the Riemann-type extensions of Dunford integral and Pettis integral, Henstock-Dunford integral and Henstock-Pettis integral. We discuss the relationships between the Henstock-Dunford integral and Dunford integral, Henstock-Pettis integral and Pettis integral. We prove the Harnack extension theorems and the convergence theorems for Henstock-Dunford and Henstock-Pettis integrals.

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1. Introduction. During 1957–1958, R. Henstock and J. Kurzweil, independently, gave a Riemann-type integral called the Henstock-Kurzweil integral (or Henstock integral) (see [7]). It is a kind of nonabsolute integral and contains the Lebesgue integral. It has been proved that this integral is equivalent to the special Denjoy integral [7]. The Dunford, Pettis integrals are generalizations of the Lebesgue integral to Banach-valued functions. In [5], R. A. Gordon gave two Denjoy-type extensions of the Dunford, Pettis integrals, the Denjoy-Dunford and Denjoy-Pettis integrals, and discussed their properties.

In this paper, we give the Riemann-type extensions of Dunford, Pettis integrals, the Henstock-Dunford, Henstock-Pettis integrals, and discuss the relationships between the Henstock-Dunford integral and Dunford integral, Henstock-Pettis integral and Pettis integral. We prove the Harnack extension theorems and the convergence theorems for Henstock-Dunford and Henstock-Pettis integrals.

Throughout this paper, $X$ denotes a real Banach space and $X^*$ its dual. $B(X^*) = \{x^* \in X^* : \|x^*\| \leq 1\}$ is the unit ball in $X^*$. $I_0 = [a, b]$ is a closed interval in $\mathbb{R}$.

We first give some preliminaries. A partition $D$ of $[a, b]$ is a finite collection of interval-point pairs $(I, t)$ with the intervals nonoverlapping and their union $[a, b]$. Here $t$ is the associated point of $I$. We write $D = \{(I, t)\}$, it is said to be $\delta$-fine partition of $[a, b]$ if for each interval-point pair $(I, t)$, we have $t \in I \subset (t - \delta(t), t + \delta(t))$.

DEFINITION 1.1 (see [7]). A function $f : [a, b] \rightarrow \mathbb{R}$ is Henstock integrable if there exists a function $F : [a, b] \rightarrow \mathbb{R}$ such that for every $\epsilon > 0$ there is a function $\delta(t) > 0$ such that for any $\delta$-fine partition $D = \{[u, v]; t\}$ of $[a, b]$, we have

$$\left| \sum (f(t)(v-u)-F(u,v)) \right| < \epsilon,$$  \hspace{1cm} (1.1)

where the sum $\sum$ is understood to be over $D = \{([u, v], t)\}$ and $F(u, v) = F(v) - F(u)$. We write $(H) \int_{I_0} f = F(I_0)$. 

The function $f$ is said to be Henstock integrable on the set $E \subseteq [a, b]$ if the function $f \chi_E$ is Henstock integrable on $[a, b]$. We write $(H) \int_0^1 f \chi_E = (H) \int_0^1 f$.

**Definition 1.2** (see [1, 5, 7]). A function $f : [a, b] \rightarrow \mathbb{R}$ is Denjoy (or special Denjoy) integrable if there exists an $ACG$ (or $ACG^*$) function $F : [a, b] \rightarrow \mathbb{R}$ such that $D_{ap}F(t) = f(t)$ (or $F'(t) = f(t)$) almost everywhere on $[a, b]$, where $D_{ap}F(t)$ denotes the approximate derivative of $F$ at $t$. We write $(D) \int_0^1 f = F(I_0)$ (or $(D^*) \int_0^1 f = F(I_0)$).

The function $f$ is said to be Denjoy (or special Denjoy) integrable on the set $E \subseteq [a, b]$ if the function $f \chi_E$ is Denjoy (or special Denjoy) integrable on $[a, b]$. We write $(D) \int_0^1 f \chi_E = (D) \int_0^1 f$ (or $(D^*) \int_0^1 f \chi_E = (D^*) \int_0^1 f$).

If $f$ is special Denjoy integrable, then $f$ is Denjoy integrable.

**Lemma 1.3** (see [7]). A function $f : [a, b] \rightarrow \mathbb{R}$ is Henstock integrable on $[a, b]$ if and only if $f$ is the special Denjoy integrable on $[a, b]$.

**Definition 1.4** (see Gordon [5]). (a) A function $f : [a, b] \rightarrow X$ is Denjoy-Dunford integrable on $[a, b]$ if for each $x^* \in X^*$ the function $x^*f$ is Denjoy integrable on $[a, b]$ and if for every interval $I$ in $[a, b]$ there exists a vector $x^*_I$ in $X^*$ such that $x^*_I(x^*) = \int_I x^* f$ for all $x^* \in X^*$. We write $x^*_I = (DD) \int_0^1 f = F(I_0)$ and $F$ is called the primitive of $f$ on $I_0$.

(b) A function $f : [a, b] \rightarrow X$ is Denjoy-Pettis integrable on $[a, b]$ if $f$ is Denjoy-Dunford integrable on $[a, b]$ and if for every interval $I$ in $[a, b]$ there exists a vector $x^*_I \in X$ for every interval $I$ in $[a, b]$. We write $x^*_I = (DP) \int_0^1 f = F(I_0)$ and $F$ is called the primitive of $f$ on $I_0$.

The function $f$ is said to be integrable in one of the above senses on the set $E \subseteq [a, b]$ if the function $f \chi_E$ is integrable in that sense on $[a, b]$.

**Lemma 1.5** (see [3]). A function $f : [a, b] \rightarrow X$ is Denjoy-Dunford integrable on $[a, b]$ if and only if $x^*f$ is Denjoy integrable on $[a, b]$ for all $x^* \in X^*$.

2. Definition and properties. In the following, we give the Riemann-type extensions of Dunford, Pettis integrals, and discuss the relationships between Henstock-Dunford integral and Dunford integral, Henstock-Pettis integral and Pettis integral.

**Definition 2.1.** (a) A function $f : [a, b] \rightarrow X$ is Henstock-Dunford integrable on $[a, b]$ if for each $x^* \in X^*$ the function $x^*f$ is Henstock integrable on $[a, b]$ and if for every interval $I$ in $[a, b]$ there exists a vector $x^*_I \in X^*$ such that $x^*_I(x^*) = \int_I x^* f$ for all $x^* \in X^*$. We write $x^*_I = (HD) \int_0^1 f = F(I_0)$ and $F$ is called the primitive of $f$ on $I_0$.

(b) A function $f : [a, b] \rightarrow X$ is Henstock-Pettis integrable on $[a, b]$ if $f$ is Henstock-Dunford integrable on $[a, b]$ and if for every interval $I$ in $[a, b]$ there exists a vector $x^*_I \in X$ for every interval $I$ in $[a, b]$. We write $x^*_I = (HP) \int_0^1 f = F(I_0)$ and $F$ is called the primitive of $f$ on $I_0$.

The function $f$ is said to be integrable in one of the above senses on the set $E \subseteq [a, b]$ if the function $f \chi_E$ is integrable in that sense on $[a, b]$.

By the above definitions and **Definition 1.4**, it is easy to see that if $f$ is Henstock-Dunford (or Henstock-Pettis) integrable on $I_0$, then $f$ is Denjoy-Dunford (or Denjoy-Pettis) integrable.
**Theorem 2.2.** A function \( f : [a, b] \to X \) is Henstock-Dunford integrable on \([a, b]\) if and only if \( x^* f \) is Henstock integrable on \([a, b]\) for all \( x^* \in X^* \).

**Proof.** If \( f \) is Henstock-Dunford integrable on \([a, b]\), for every \( x^* \in X^* \), by Definition 2.1, \( x^* f \) is Henstock integrable on \([a, b]\). Conversely, if \( x^* f \) is Henstock integrable on \([a, b]\), it follows from Lemma 1.3 that \( x^* f \) is Denjoy integrable on \([a, b]\) and \( (D) \int_a^b x^* f = (H) \int_a^b x^* f \). It follows from Lemma 1.5 that \( f \) is Denjoy-Dunford integrable on \([a, b]\), and for every interval \( I \) in \([a, b]\) there exists a vector \( x^*_I \) in \( X^* \) such that \( x^*_I (x^*) = (D) \int_I x^* f \) for all \( x^* \) in \( X^* \), that is, \( x^*_I (x^*) = (H) \int_I x^* f \) for all \( x^* \) in \( X^* \). So \( f \) is Henstock-Dunford integrable on \([a, b]\). \( \square \)

**Theorem 2.3.** If the function \( f : [a, b] \to X \) is Henstock-Dunford integrable on \([a, b]\), then each perfect set in \([a, b]\) contains a portion on which \( f \) is Dunford integrable.

**Proof.** Since the function \( f : [a, b] \to X \) is Henstock-Dunford integrable on \([a, b]\), then for each \( x^* \in X^* \), \( x^* f \) is Henstock integrable on \([a, b]\). It follows from [8] that each perfect set in \([a, b]\) contains a portion on which \( x^* f \) is Lebesgue integrable. So \( f \) is Dunford integrable on a portion. \( \square \)

**Theorem 2.4.** If the function \( f : [a, b] \to X \) is Henstock-Dunford integrable on \([a, b]\), then there is a sequence \( \{X_k\} \) of closed subsets such that \( X_k \subset X_{k+1} \) for all \( k \), \( \bigcup_{k=1}^\infty X_k = [a, b] \), \( f \) is Dunford integrable on each \( X_k \) and

\[
\lim_{k \to \infty} (\text{Dunford}) \int_{X_k \cap [a, x]} f(t) \, dt = (HD) \int_a^x f(t) \, dt \quad \text{weakly} \quad (2.1)
\]

uniformly on \([a, b]\).

**Proof.** It follows from Theorem 2.2 that a function \( f : [a, b] \to X \) is Henstock-Dunford integrable on \([a, b]\) if and only if \( x^* f \) is Henstock integrable on \([a, b]\) for all \( x^* \in X^* \). From [8], \( x^* f \) is Henstock integrable on \([a, b]\), then there is a sequence \( \{X_k\} \) of closed subsets such that \( X_k \subset X_{k+1} \) for all \( k \), \( \bigcup_{k=1}^\infty X_k = [a, b] \), \( x^* f \) is Lebesgue integrable on each \( X_k \) and

\[
\lim_{k \to \infty} (L) \int_{X_k \cap [a, x]} x^* f(t) \, dt = (H) \int_a^x x^* f(t) \, dt \quad (2.2)
\]

uniformly on \([a, b]\) for each \( x^* \in X^* \). So we obtain that \( f \) is Dunford integrable on each \( X_k \) and

\[
\lim_{k \to \infty} (\text{Dunford}) \int_{X_k \cap [a, x]} f(t) \, dt = (HD) \int_a^x f(t) \, dt \quad \text{weakly} \quad (2.3)
\]

uniformly on \([a, b]\).

**Theorem 2.5.** If the function \( f : [a, b] \to X \) is Henstock-Dunford integrable on \([a, b]\), then there exists a sequence \( \{X_k\} \) of closed sets, \( \bigcup_{k=1}^\infty X_k = [a, b] \), \( f \) is Dunford integrable on each \( X_k \).

**Proof.** Since \( f \) Henstock-Dunford integrable on \([a, b]\), by Definition 2.1, for each \( x^* \in X^* \), \( x^* f \) is Henstock integrable on \([a, b]\), and for every interval \( I \subset [a, b] \),
\[ \int x^* f = x^* \int f, \] and \( F(I) = \int f \in X. \) Since \( x^* f \) is Henstock integrable, then \( x^* F \) is \( ACG^* \). So there is a sequence \( \{ X_k \} \) of closed subsets such that \( \bigcup_{k=1}^{\infty} X_k = [a, b] \) and \( x^* F \) is \( VB^* \) on each \( X_k \). From [7, Lemma 6.18], \( x^* f \) is Lebesgue integrable on each \( X_k \). So we obtain that \( f \) is Dunford integrable on each \( X_k \).

**Theorem 2.6.** Suppose that \( X \) contains no copy of \( c_0 \) and \( f : [a, b] \to X. \) If the function \( f \) is Henstock-Pettis integrable on \([a, b]\), then each perfect set in \([a, b]\) contains a portion on which \( f \) is Pettis integrable.

**Proof.** Since the function \( f : [a, b] \to X \) is Henstock-Pettis integrable on \([a, b]\), then \( f \) is Denjoy-Pettis integrable on \([a, b]\). It follows from [5, Theorem 38] that each perfect set in \([a, b]\) contains a portion on which \( f \) is Pettis integrable.

In the fact, from [3, Theorem 10], we have that if each Henstock-Pettis integrable function defined on \([a, b]\) is Pettis integrable on a portion of every close set, then \( X \) does not contain \( c_0 \).

**Theorem 2.7.** Suppose that \( X \) contains no copy of \( c_0 \) and \( f : [a, b] \to X \) is measurable. If the function \( f : [a, b] \to X \) is Henstock-Pettis integrable on \([a, b]\), then there exists a sequence \( \{ X_k \} \) of closed sets such that for each \( x^* \in X^* \), \( f \) is Pettis integrable on each \( X_k \), and

\[ \lim_{k \to \infty} (\text{Pettis}) \int_{X_k} f = (HP) \int_a^b f \text{ weakly.} \] \hspace{1cm} (2.4)

**Proof.** Since \( f \) is Henstock-Pettis integrable on \([a, b]\), and by Definition 2.1, for each \( x^* \in X^* \), \( x^* F \) is Henstock integrable on \([a, b]\), and for every interval \( I \subset [a, b]\),

\[ \int_I x^* f = x^* \int f, \] and \( F(I) = \int f \in X. \) Since \( x^* f \) is Henstock integrable, then \( x^* F \) is \( ACG^* \). So there is a sequence \( \{ X_k \} \) of closed subsets such that \( \bigcup_{k=1}^{\infty} X_k = [a, b] \) and \( x^* F \) is \( VB^* \) on each \( X_k \). From [7, Lemma 6.18], \( x^* f \) is Lebesgue integrable on each \( X_k \). So we obtain that \( f \) is Dunford integrable on each \( X_k \).

**Theorem 2.8.** Suppose that \( X \) contains no copy of \( c_0 \). If the function \( f : [a, b] \to X \) is Henstock-Pettis integrable on \([a, b]\), then there exists a sequence \( \{ X_k \} \) of closed sets, \( \bigcup_{k=1}^{\infty} X_k = [a, b] \), \( f \) is Pettis integrable on each \( X_k \).

**Proof.** Since \( f \) is Henstock-Pettis integrable on \([a, b]\), by Definition 2.1, for each \( x^* \in X^* \), \( x^* f \) is Henstock integrable on \([a, b]\), and for every interval \( I \subset [a, b]\),
\[ \int_{I} x^{*} f = x^{*} \int_{I} f, \] and \( F(I) = \int_{I} f \in X \). Since \( x^{*} f \) is Henstock integrable, then \( x^{*} F \) is \( ACG^{*} \). So there is a sequence \( \{X_k\} \) of closed subsets such that \( \bigcup_{k=1}^{\infty} X_k = [a, b] \) and \( x^{*} F \) is \( VB^{*} \) on each \( X_k \). For each \( k \in N \), let \( (a, b) - X_k = \bigcup_{n=1}^{\infty} (c_{kn}^{h}, d_{kn}^{h}) \). Then

\[ \sum_{n=1}^{\infty} |x^{*} \int_{c_{kn}^{h}}^{d_{kn}^{h}} f| < \infty. \]  

(2.8)

Since \( X \) contains no copy of \( c_0 \), by Bessaga-Pelczynski theorem [2, page 22], \( \sum_{n=1}^{\infty} \int_{c_{kn}^{h}}^{d_{kn}^{h}} f \) is unconditionally convergent in norm. Also

\[ \sum_{n=1}^{\infty} \sup_{[a_{kn}^{h}, b_{kn}^{h}] \subset [a_{kn}^{h}, d_{kn}^{h}]} \left| x^{*} \int_{a_{kn}^{h}}^{b_{kn}^{h}} f \right| < \infty. \]  

(2.9)

By Harnack extension theorem [7, page 41], we have

\[ \int_{X_k} x^{*} f = \int_{a}^{b} x^{*} f - \sum_{n=1}^{\infty} \int_{c_{kn}^{h}}^{d_{kn}^{h}} x^{*} f = x^{*} \left( \int_{a}^{b} f - \sum_{n=1}^{\infty} \int_{c_{kn}^{h}}^{d_{kn}^{h}} f \right). \]  

(2.10)

Hence \( \int_{X_k} f = \int_{a}^{b} f - \sum_{n=1}^{\infty} \int_{c_{kn}^{h}}^{d_{kn}^{h}} f \in X \) and \( \int_{X_k} x^{*} f = x^{*} \int_{X_k} f \).

So, for every closed set \( H \subset X_k \), we have \( \int_{H} x^{*} f = x^{*} \int_{H} f \) and \( \int_{H} f \in X \). Since \( \int_{a}^{b} f \chi_{X_k} = \int_{X_k} f \in X \), \( \int_{a}^{b} f \chi_{H} = \int_{H} f \in X \), then for every closed interval \( I \subset [a, b] \), \( \int_{I} f \chi_{X_k} = \int_{I \cap X_k} f \in X \). By [5, Theorem 23, page 79], \( f \chi_{X_k} \) is Pettis integrable on \([a, b]\), that is, \( f \) is Pettis integrable on each \( X_k \).

\[ \boxed{\text{Hence} \int_{X_k} f = \\ \int_{a}^{b} f - \sum_{n=1}^{\infty} \int_{c_{kn}^{h}}^{d_{kn}^{h}} f \in X \text{ and } \int_{X_k} x^{*} f = x^{*} \int_{X_k} f.} \]  

3. The extension theorems and convergence theorems. Now we consider the extension theorems and convergence theorems of the Henstock-Dunford and Henstock-Pettis integrals.

**Theorem 3.1.** Let \( E \) be a closed subset in \([a, b]\) and \((a, b) - E\) the union of \( \{(a_k, b_k)\} \), \( k = 1, 2, \ldots \). If \( f : [a, b] \to X \) is Henstock-Dunford integrable on \( E \) and each interval \([a_k, b_k]\) with

\[ \sum_{k=1}^{\infty} \omega\left( \int_{a_k}^{b} x^{*} f, [a_k, b_k] \right) < \infty \]  

(3.1)

for each \( x^{*} \in X^{*} \), then \( f \) is Henstock-Dunford integrable on \([a, b]\) and

\[ \left< x^{*}, (HD) \int_{a}^{b} f \right> = \left< x^{*}, (HD) \int_{a}^{b} f \chi_{E} \right> + \sum_{k=1}^{\infty} \left< x^{*}, (HD) \int_{a_k}^{b_k} f \right> \]  

(3.2)

for each \( x^{*} \in X^{*} \).

**Proof.** From the conditions of **Theorem 3.1**, we have the function \( x^{*} f \) satisfies the hypothesis of [7, Corollary 7.11]. So we have \( x^{*} f \) is Henstock integrable on \([a, b]\) and

\[ (H) \int_{a}^{b} x^{*} f = (H) \int_{a}^{b} x^{*} f \chi_{E} + \sum_{k=1}^{\infty} (H) \int_{a_k}^{b_k} x^{*} f. \]  

(3.3)
It follows from Theorem 2.2 that \( f \) is Henstock-Dunford integrable on \([a,b]\) and the above equality means that

\[
\left\langle x^*, (HD) \int_a^b f \right\rangle = \left\langle x^*, (HD) \int_a^b f \chi_E \right\rangle + \sum_{k=1}^{\infty} \left\langle x^*, (HD) \int_{a_k}^{b_k} f \right\rangle
\]

(3.4)

for each \( x^* \in X^* \).

**Theorem 3.2.** Let \( E \) be a closed subset in \([a,b]\) and \( \{ (a_k, b_k) \} \) be an enumeration of the intervals contiguous to \( E \) in \((a,b)\). Suppose that \( f : [a,b] \to X \) is Henstock-Pettis integrable on \( E \) and each interval \([a_k, b_k]\). If \( \sum_{k=1}^{\infty} \omega(\int_{a_k}^{b_k} x^* f, (a_k, b_k)) < \infty \) for each \( x^* \in X^* \) and the series \( \sum_{k=1}^{\infty} (HP) \int_{[a_k, b_k] \cap J} f \) is unconditionally convergent for every subinterval \( J \) of \([a,b]\), then \( f \) is Henstock-Pettis integrable on \([a,b]\) and

\[
(HP) \int_a^b f = (HP) \int_a^b f \chi_E + \sum_{k=1}^{\infty} (HP) \int_{a_k}^{b_k} f.
\]

(3.5)

**Proof.** From Theorem 3.1, we have the function \( f \) is Henstock-Dunford integrable on \([a,b]\) and \((H) \int_a^b x^* f = (H) \int_a^b x^* f \chi_E + \sum_{k=1}^{\infty} (H) \int_{a_k}^{b_k} x^* f \). To show that \( f \) is in fact Henstock-Pettis integrable on \([a,b]\), we need to show that \((HD) \int_{J} f \) belongs to \( X \) for each closed interval \( J \) in \([a,b]\).

Let \( E_0 = E \cap J \). Then \( E_0 \) is a closed set. Since \( f \chi_E \) is Henstock-Pettis integrable on \( J \), then \( f \chi_{E_0} \) is Henstock-Pettis integrable on \( J \), that is, \( f \) is Henstock-Pettis integrable on \( E_0 \). And \( \{ (a_k, b_k) \cap J \} \) is an enumeration of the intervals contiguous to \( E_0 \) in \( J \), so \( f \) is Henstock-Pettis integrable on them and \( \sum_{k=1}^{\infty} (HP) \int_{[a_k, b_k] \cap J} f \) is an unconditionally convergent series in \( X \). Now, if we apply Theorem 3.1 to \( E_0 \) in \( J \), we get

\[
\left\langle x^*, (HD) \int_{J} f \right\rangle = \left\langle x^*, (HP) \int_{J} f \chi_{E_0} \right\rangle + \sum_{k=1}^{\infty} \left\langle x^*, (HP) \int_{[a_k, b_k] \cap J} f \right\rangle
\]

(3.6)

for each \( x^* \in X^* \), that is,

\[
\left\langle x^*, (HD) \int_{J} f \right\rangle = \left\langle x^*, (HP) \int_{J} f \chi_{E_0} + \sum_{k=1}^{\infty} (HP) \int_{[a_k, b_k] \cap J} f \right\rangle
\]

(3.7)

for each \( x^* \in X^* \). We conclude that

\[
(HP) \int_a^b f = (HP) \int_a^b f \chi_{E_0} + \sum_{k=1}^{\infty} (HP) \int_{[a_k, b_k] \cap J} f.
\]

(3.8)

Hence, \( f \) is Henstock-Pettis integrable on \([a,b]\) and

\[
(HP) \int_a^b f = (HP) \int_a^b f \chi_{E_0} + \sum_{k=1}^{\infty} (HP) \int_{[a_k, b_k] \cap J} f.
\]

(3.9)

**Corollary 3.3.** Suppose that \( X \) contains no copy of \( c_0 \). Let \( E \) be a closed subset in \([a,b]\) and \( \{ (a_k, b_k) \} \) be an enumeration of the intervals contiguous to \( E \) in \((a,b)\). Suppose that \( f : [a,b] \to X \) is Henstock-Pettis integrable on \( E \) and each interval \([a_k, b_k]\).
If \( \sum_{k=1}^{\infty} \omega([a_k, b_k]) < \infty \) for each \( x^* \in X^* \), then \( f \) is Henstock-Pettis integrable on \([a, b]\) and
\[
(HP) \int_{a}^{b} f = (HP) \int_{a}^{b} f \chi_E + \sum_{k=1}^{\infty} (HP) \int_{a_k}^{b_k} f. \tag{3.10}
\]

**Theorem 3.4.** Suppose that \( X \) is weakly sequentially complete and \( f : [a, b] \to X \) is Henstock-Dunford integrable on \([a, b]\). If \( f \) is measurable, then \( f \) is Henstock-Pettis integrable on \([a, b]\).

**Proof.** It is similar to the proof of [5, Theorem 40]. \( \square \)

**Lemma 3.5** (see [1, 5]). Let \( \Gamma \) be a family of open intervals in \((a, b)\) and suppose that \( \Gamma \) has the following properties:
1. if \((\alpha, \beta)\) and \((\beta, \gamma)\) belong to \( \Gamma \), then \((\alpha, \gamma)\) belongs to \( \Gamma \);
2. if \((\alpha, \beta)\) belong to \( \Gamma \), then every open interval in \((\alpha, \beta)\) belongs to \( \Gamma \);
3. if all of the intervals contiguous to the perfect set \( E \subset [a, b] \) belong to \( \Gamma \), then there exists an interval \( I \) in \( \Gamma \) such that \( I \cap E \neq \emptyset \).
Then \( \Gamma \) contains the interval \((a, b)\).

**Lemma 3.6.** Suppose that \( f_n : [a, b] \to \mathbb{R}, f : [a, b] \to \mathbb{R} \), and
1. \( f_n \to f \) almost everywhere on \([a, b]\) as \( n \to \infty \), where each \( f_n \) is Henstock (or \( D^* \)) integrable on \([a, b]\);
2. the primitives \( F_n \) of \( f_n \) are continuous uniformly in \( n \) and \( ACG^* \) uniformly in \( n \).
Then \( f \) is Henstock (or \( D^* \)) integrable on \([a, b]\) and
\[
\lim_{n \to \infty} \int_{a}^{b} f_n = \int_{a}^{b} f. \tag{3.11}
\]

**Definition 3.7.** Let \( F : [a, b] \to X \) and let \( E \) be a subset of \([a, b]\).
(a) \( F \) is \( BV^* \) on \( E \) if \( \sup \{ \sum_i \omega(F; [c_i, d_i]) \} \) is finite, where the supremum is taken over all finite collections \([c_i, d_i]\) of nonoverlapping intervals that have endpoints in \( E \), \( \omega \) denotes the oscillation of \( F \) over \([c_i, d_i]\), that is,
\[
\omega(F; [c_i, d_i]) = \sup \{ ||F(x) - F(y)|| ; x, y \in [c_i, d_i] \}. \tag{3.12}
\]
(b) \( F \) is \( AC^* \) on \( E \) if for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( \sum_i \omega(F; [c_i, d_i]) < \epsilon \) whenever \([c_i, d_i]\) is a finite collection of nonoverlapping intervals that have endpoints in \( E \) and satisfy \( \sum_i (d_i - c_i) < \delta \).
(c) \( F \) is \( BVG^* \) on \( E \) if \( E \) can be expressed as a countable union of sets on each of which \( F \) is \( BV^* \).
(d) \( F \) is \( ACG^* \) on \( E \) if \( F \) is continuous on \( E \) and if \( E \) can be expressed as a countable union of sets on each of which \( F \) is \( AC^* \).

**Theorem 3.8.** Suppose that \( X \) is weakly sequentially complete and
1. \( f_n \to f \) weakly almost everywhere on \([a, b]\) as \( n \to \infty \), where each \( f_n \) is Henstock-Pettis integrable on \([a, b]\);
2. the primitives \( F_n \) of \( f_n \) are continuous uniformly in \( n \) and \( ACG^* \) uniformly in \( n \).
Then $f$ is Henstock-Pettis integrable on $[a, b]$ and
\[
\lim_{n \to \infty} \int_a^b f_n = \int_a^b f \text{ weakly.} \tag{3.13}
\]

**Proof.** Let
\[
\Gamma = \left\{ (\alpha, \beta) \in [a, b] : f \text{ is Henstock-Pettis integrable on } [\alpha, \beta], \int_{\alpha}^{\beta} f_n \to \int_{\alpha}^{\beta} f \text{ weakly} \right\}.
\]

We must show that $\Gamma$ contains $(a, b)$ and by Lemma 3.5 it is sufficient to verify that $\Gamma$ satisfies Romanovski’s four conditions.

Conditions (1) and (2) are easily verified.

Suppose that $(\alpha, \beta)$ belongs to $\Gamma$ for every interval $[\alpha, \beta]$ in $(c, d)$. For each positive integer $n > 2/(d - c)$, define $I_n = (c + 1/n, d - 1/n)$ and let $x_n = x_{n}^{**}$.

Then we have
\[
x^{**}_{(c,d)}(x^*) = \int_c^d x^* f = \lim_{n \to \infty} \int_{I_n} x^* f = \lim_{n \to \infty} x_n^{*}(x_n) \tag{3.15}
\]
for each $x^*$ in $X^*$. Since $X$ is weakly sequentially complete, the sequence $\{x_n\}$ converges weakly to an element $x_0$ of $X$ and we must have $x^{**}_{(c,d)} = x_0$. It follows easily that $(c, d)$ belongs to $\Gamma$ and this verifies condition (3).

Now let $E$ be a perfect set in $[a, b]$ such that each of the intervals in $[a, b]$ contiguous to $E$ belongs to $\Gamma$.

Since $\{F_n\}$ is continuous uniformly in $n$ and $ACG^*$ uniformly in $n$, then for each $x^* \in X^*$, $\{x^* F_n\}$ is continuous uniformly in $n$ and $ACG^*$ uniformly in $n$, and $x^* f_n \to x^* f$ almost everywhere in $[a, b]$. It follows from [1] that $x^* f$ is special Denjoy integrable on $[a, b]$. So there exists an interval $[u, v] \subset P$ and $\{k_n\}_n$ such that $\{F_n\}$ is $AC^*$ uniformly in $n$ on $P = E \cap (u, v)$ and the series $\sum_k \omega(F_n; [u_k, v_k])$ unconditionally converges where $(u, v) - E = \cup_k (u_k, v_k)$. Since $\sum_k \omega(\int_{u_k}^{v_k} x^* f_n; [u_k, v_k]) < \infty$ for each $x^* \in X^*$. By Corollary 3.3, we have
\[
\int_u^v f_n = \int_p f_n + \sum_k \int_{u_k}^{v_k} f_n. \tag{3.16}
\]

It follows from [4, Theorem 3] that $f$ is Pettis integrable on $P$ and $\int_p f_n \to \int_p f$ weakly.

Since $\{F_n\}$ is $AC^*$ uniformly in $n$ on $P$, so for every $\epsilon > 0$ there exists $N$ such that $\sum_{k \geq N} \parallel \int_{u_k}^{v_k} f_n \parallel < \epsilon$, $n = 1, 2, \ldots$. For every $x^* \in B(X^*)$, we have $\sum_{k \geq N} \parallel \int_{u_k}^{v_k} x^* f_n \parallel < \epsilon$, $n = 1, 2, \ldots$. So $\sum_{k \geq N} \parallel \int_{u_k}^{v_k} x^* f \parallel < \epsilon$. Since $X$ is weakly sequentially complete and $X$ does not contain $c_0$, hence $\sum_{k \geq N} \int_{u_k}^{v_k} f$ unconditionally converges. By (3.16),
\[
x^* \int_u^v f_n = x^* \int_p f_n + x^* \sum_k \int_{u_k}^{v_k} f_n. \tag{3.18}
\]
Let \( n \to \infty \), we have
\[
x^{**}(u,v) = x^* \int_p f + x^* \sum_k \int_{u_k}^v f.
\]
Hence
\[
x^{**}(u,v) = \int_p f + \sum_k \int_{u_k}^v f \in X,
\]
that is, \( f \) is Henstock-Pettis integrable on \([u,v]\). So \((u,v) \in \Gamma\). This shows that \((u,v)\) belongs to \(\Gamma\) and \(\Gamma\) satisfies condition (4). This completes the proof.

**Theorem 3.9.** Suppose that \( X \) is weakly sequentially complete and \( f_n \to f \) weakly almost everywhere on \([a,b]\) as \( n \to \infty \), where each \( f_n \) is Henstock-Pettis integrable on \([a,b]\). If there is a scalar function \( g \) with \( \| f_n(\cdot) \| \leq g(\cdot) \) almost everywhere for all \( n \) and if \( g < \infty \), then \( f \) is Henstock-Pettis integrable on \([a,b]\) and
\[
\lim_{n \to \infty} \int_a^b f_n = \int_a^b f \text{ weakly.}
\]

**Proof.** It is similar to the proof of Theorem 3.8.

**Definition 3.10.** Let \( \{f_n\} \) be a family of Henstock-Pettis integrable functions defined on \([a,b]\). The family \( \{x^* f_n : x^* \in B(X^*)\} \) is uniformly integrable in the generalized sense on \([a,b]\), if for each perfect set \( E \subset [a,b] \) there exists an interval \([c,d] \subset [a,b]\) with \( c,d \in E \) and \( E \cap (c,d) \neq \emptyset \) such that \( \{x^* f_n : x^* \in B(X^*)\} \) is uniformly integrable on \( P = E \cap (c,d) \) and for every \( \alpha \) the series \( \sum_k \int_{c_k}^{d_k} f_n \) is unconditionally convergent where \( (c,d) - E = \bigcup k(c_k,d_k) \).

**Theorem 3.11.** Suppose that \( X \) is weakly sequentially complete and

1. \( f_n \to f \) weakly almost everywhere on \([a,b]\) as \( n \to \infty \), where each \( f_n \) is Henstock-Pettis integrable on \([a,b]\).
2. The family \( \{x^* f_n : x^* \in B(X^*) \} \) is uniformly integrable in the generalized sense on \([a,b]\).
3. For each \( x^* \in X^* \), \( \lim_{n \to \infty} \int_{c}^{d} x^* f_n = \int_{c}^{d} x^* f \) uniformly for every \([c,d] \subset [a,b]\).

Then \( f \) is Henstock-Pettis integrable on \([a,b]\) and
\[
\lim_{n \to \infty} \int_a^b f_n = \int_a^b f \text{ weakly.}
\]

**Proof.** It is similar to the proof of Theorem 3.8. The only difference is that the family \( \{x^* f_n : x^* \in B(X^*) \} \) is uniformly integrable in the generalized sense on \([a,b]\), then there is a portion \( P = E \cap I \) of \( E \) such that the family \( \{x^* f_n X|E| \} \) is uniformly integrable on \( P \). So \( f \) is Pettis integrable on \( P \).

**Theorem 3.12.** Suppose that \( X \) is weakly sequentially complete and

1. \( f_n \to f \) weakly almost everywhere on \([a,b]\) as \( n \to \infty \), where each \( f_n \) is Henstock-Pettis integrable on \([a,b]\) and \( f \) is measurable,
2. the primitives \( F_n \) of \( f_n \) are weakly continuous uniformly in \( n \) and weakly ACG* uniformly in \( n \), that is, for every \( x^* \in X^* \), \( x^* F_n \) are continuous uniformly in \( n \) and ACG* uniformly in \( n \).
Then $f$ is Henstock-Pettis integrable on $[a,b]$ and
\[ \lim_{n \to \infty} \int_a^b f_n = \int_a^b f \text{ weakly.} \] (3.23)

**Proof.** For each $x^*$ in $X^*$, we have
(1) $x^* f_n \to x^* f$ almost everywhere on $[a,b]$ as $n \to \infty$, where each $x^* f_n$ is Henstock integrable on $[a,b]$,
(2) the primitives $x^* F_n$ of $x^* f_n$ are continuous uniformly in $n$ and ACG* uniformly in $n$. It follows from Lemma 3.6 that $x^* f$ is Henstock integrable on $[a,b]$ and
\[ \int_a^b x^* f_n \to \int_a^b x^* f \text{ as } n \to \infty. \] (3.24)

By Theorem 2.2, $f$ is Henstock-Dunford integrable on $[a,b]$. Since $X$ is weakly sequentially complete and $f$ is measurable, by Theorem 3.4, $f$ is Henstock-Pettis integrable on $[a,b]$. \qed

**Theorem 3.13.** Suppose that the unit ball $B(X^*)$ of $X^*$ is weak* sequentially compact and
(1) $f_n \to f$ weakly almost everywhere in $[a,b]$ as $n \to \infty$, where each $f_n$ is Henstock-Pettis integrable on $[a,b]$,
(2) the primitives $F_n$ of $f_n$ are continuous uniformly in $n$ and ACG* uniformly in $n$. Then $f$ is Henstock-Pettis integrable on $[a,b]$ and
\[ \lim_{n \to \infty} \int_a^b f_n = \int_a^b f \text{ weakly.} \] (3.25)

**Proof.** Suppose that $I \subset I_0$. Let $C$ be the weak closure of $\{\int_I f_n : n \in \mathbb{N}\}$. For each $x^*$ in $X^*$, $\{x^* F_n : n \in \mathbb{N}\}$ is continuous uniformly in $n$ and ACG* uniformly in $n$ in $[a,b]$, and further $\int_a^b x^* f_n = x^* \int_a^b f_n$. A convergence theorem, namely Lemma 3.6, guarantees that $x^* f$ is Henstock integrable on $[a,b]$ and $\lim_{n \to \infty} \int_a^b x^* f_n = \int_a^b x^* f$ for each $x^*$ in $X^*$. We observe that $C$ is bounded and that $C - \{\int_I f_n : n \in \mathbb{N}\}$ contains at most one point. We will prove that $C$ is weakly compact.

Suppose that $C$ is not weakly compact. An appeal to a theorem of James [6, Theorem 1] produces a bounded sequence $(x^*_k)$ in $X^*$, a sequence $(x_n)$ in $C$, and an $\epsilon > 0$ such that $x^*_k (x_n) = 0$ for $k > n$ and $x^*_k (x_n) > \epsilon$ for $n \geq k$. By passing to subsequences and relabelling, we can find a subsequence $(\int_I g_n)$ of $(\int_I f_n)$ and a subsequence $(y^*_k)$ of $x^*_k$ such that
\[
\begin{align*}
\int_I g_n &= \int_I y^*_k g_n = 0 \quad \text{for } k > n, \\
\int_I g_n &= \int_I y^*_k g_n > \epsilon \quad \text{for } n \geq k, \\
\lim_{n \to \infty} \int_I x^* g_n &= \int_I x^* f \quad \forall x^* \text{ in } X^*.
\end{align*}
\] (3.26)

Since the unit ball $B(X^*)$ of $X^*$ is weak* sequentially compact, the sequence $(y^*_k)$ has a subsequence $(y^*_k)$ which weak* converges to $y^*_0$, so $\lim_{j \to \infty} y^*_k f = y^*_0 f$ on $I_0$. \qed
\[
\lim_{j \to \infty} y^*_j F = y^*_0 F \text{ on } I_0, \text{ that is, } \lim_{j \to \infty} \int_{I_j} y^*_j f = \int_I y^*_0 f. \text{ To force a contradiction, note that for each } k, \lim_{n \to \infty} \int_{I_k} y^*_k f_n = \int_I y^*_0 f. \text{ Hence } \int_{I_k} y^*_k f \geq \epsilon \text{ for each } k, \text{ and } \int_I y^*_0 f \geq \epsilon. \text{ On the other hand, notice that since each } g_n \text{ is Henstock-Pettis integrable, } (y^*_{kj}) \text{ weak* converges to } y^*_0, \text{ hence}
\]

\[
\lim_{j \to \infty} \int_{I_j} y^*_j g_n = \lim_{j \to \infty} y^*_j \int_{I_j} g_n = \int_I y^*_0 g_n.
\]

(3.27)

Since this holds for each \( n \), and since \( \lim_{n \to \infty} \int_I y^*_0 g_n = \int_I y^*_0 f \), we see that \( \int_I y^*_0 f \) exists weakly in \( X \). Denote \( F(I) = \int_I f \), then \( x^* F(I) = \int_I x^* f \) for each \( x^* \) in \( X^* \). So \( f \) is Henstock-Pettis integrable on \([a, b]\) and

\[
\lim_{n \to \infty} \int_a^b f_n = \int_a^b f \text{ weakly}. \tag{3.28}
\]

**Corollary 3.14.** Suppose that \( X \) is a reflexive Banach space and

1. \( f_n \to f \) weakly almost everywhere on \([a, b]\) as \( n \to \infty \), where each \( f_n \) is Henstock-Pettis integrable on \([a, b]\),

2. the primitives \( F_n \) of \( f_n \) are weakly continuous uniformly in \( n \) and weakly \( ACG^* \) uniformly in \( n \) on \([a, b]\).

Then \( f \) is Henstock-Pettis integrable on \([a, b]\) and

\[
\lim_{n \to \infty} \int_a^b f_n = \int_a^b f \text{ weakly.} \tag{3.29}
\]

**Theorem 3.15.** If the following conditions are satisfied:

1. \( \lim_{n \to \infty} f_n = f \) weakly almost everywhere on \([a, b]\) as \( n \to \infty \), where each \( f_n \) is Henstock-Dunford integrable on \([a, b]\),

2. the primitives \( F_n \) of \( f_n \) are weakly continuous uniformly in \( n \) and weakly \( ACG^* \) uniformly in \( n \).

Then \( f \) is Henstock-Dunford integrable on \([a, b]\) and

\[
\lim_{n \to \infty} \int_a^b f_n = \int_a^b f \text{ weakly.} \tag{3.30}
\]

**Proof.** Since

1. \( \lim_{n \to \infty} x^* f_n = x^* f \) almost everywhere on \([a, b]\),

2. the primitives \( x^* F_n \) of \( x^* f_n \) are continuous uniformly in \( n \) and \( ACG^* \) uniformly in \( n \).

Then, as in the proof of Theorem 3.12, \( x^* f \) is Henstock integrable on \([a, b]\) and

\[
\lim_{n \to \infty} \int_a^b x^* f_n = \int_a^b x^* f. \tag{3.31}
\]

By Theorem 2.2, \( f \) is Henstock-Dunford integrable on \([a, b]\) and

\[
\lim_{n \to \infty} \int_a^b f_n = \int_a^b f \text{ weakly.} \tag{3.32}
\]
References


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