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SOME FIXED POINT THEOREMS IN METRIC AND 2-METRIC SPACES

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Abstract. We introduce the notion of compatibility for a pair of self-maps on a 2-metric space and we have fixed point theorems for pairs as well as quadruples of self-maps on a 2-metric space satisfying certain generalised contraction conditions. Further metric space versions of the same have also been obtained.

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Brian Fisher [1] proved the following result.

Theorem 1. *Let f be a self-map on a complete metric space (M,ρ) such that*

$$\rho^2(fx,fy) \le \alpha\rho(x,fx)\rho(y,fy)+\beta\rho(x,fy)\rho(y,fx) \tag{1}$$

for all x,y in M for some nonnegative constants α, β with $\alpha<1$. Then f has a fixed point in M. If further $\beta<1$, then f has a unique fixed point in M.

In this paper we first obtain generalisations of the existence part of the 2-metric space version of Theorem 1 for a pair of self-maps and the uniqueness part of the same for four self-maps on a 2-metric space. Next we state without proof the metric space versions of some of these results. We also give a number of examples to throw light on the results discussed and the concept of compatibility of a pair of self-maps on a 2-metric space introduced here.

Recall some basic notions and facts for the sake of completeness.

Definition 2. Let X be a nonempty set. A real-valued function d on $X\times X\times X$ is said to be a 2-metric on X if

(i) given distinct elements x, y of X, there exists an element z of X such that $d(x,y,z)\ne 0$,
(ii) $d(x,y,z)=0$ when at least two of x, y, z are equal,
(iii) $d(x,y,z)=d(x,z,y)=d(y,z,x)$ for all x, y, z in X, and
(iv) $d(x,y,z)\le d(x,y,w)+d(x,w,z)+d(w,y,z)$ for all x,y,z,w in X.

When d is a 2-metric on X, the ordered pair (X,d) is called a 2-metric space.

Definition 3. A sequence $\{x_n\}$ in a 2-metric space (X,d) is said to be

(i) convergent with limit x in X if $\lim_{n\to\infty} d(x_n,x,a)=0$ for all a in X,
(ii) Cauchy if $\lim_{m,n\to\infty} d(x_n,x_m,a)=0$ for all a in X.

Definition 4. A 2-metric space is said to be complete if every Cauchy sequence in it is convergent.

Definition 5. A 2-metric on a set X is said to be continuous on X if it is sequentially continuous in two of its arguments.

It is known that a 2-metric is a nonnegative real-valued function, that it is sequentially continuous in anyone of its arguments and that if it is sequentially continuous in two of its arguments then it is sequentially continuous in all the three arguments. It was observed by Naidu and Prasad that (i) a convergent sequence in a 2-metric space need not be Cauchy (see [4, Remark 0.1 and Example 0.1]) (ii) in a 2-metric space (X,d) every convergent sequence is Cauchy if d is continuous on X [4, Remark 0.2] and (iii) the converse of (ii) is false [4, Remark 0.2 and Example 0.2].

Throughout this paper, unless otherwise stated, (X,d) is a 2-metric space; (M,ρ) is a metric space; $\mathbb{R}$ is the set of all real numbers; $\mathbb{R}^+$ is the set of all nonnegative real numbers; for a self-map θ on $\mathbb{R}^+$, θ^1 stands for θ and for a positive integer n, θ^{n+1} is the composite of θ and θ^n; φ is a monotonically increasing map from $\mathbb{R}^+$ to $\mathbb{R}^+$ with $\sum_{n=1}^{\infty}\sqrt{\varphi^n(t)} < +\infty$ for all t in $\mathbb{R}^+$; ψ is a map from $\mathbb{R}^+$ to $\mathbb{R}^+$ with $\psi(0) = 0$; K is an absolute nonnegative real constant; and, depending upon the context, f, g, S, T are self-maps on X or M.

We note that $\varphi(t) < t$ for all t in $(0,\infty)$ and that $\varphi(0) = \varphi(0+) = 0$.

Remark 6. For a monotonically increasing nonnegative real-valued function θ on $\mathbb{R}^+$ the condition "$\sum_{n=1}^{\infty}\sqrt{\theta^n(t)} < +\infty$ for all t in $\mathbb{R}^+$" neither implies nor is implied by the condition "$\theta(t+) < t$ for all t in $(0,\infty)$." Examples 7 and 8 illustrate this.

Example 7. Define $\theta:\mathbb{R}^+ \to \mathbb{R}^+$ as $\theta(t) = t^2$ if $0 \le t \le 3/4$ and $\theta(t) = 3/4$ if $t > 3/4$. Then θ is monotonically increasing on $\mathbb{R}^+$. For a positive integer n, we have $\theta^n(t) = (3/4)^{2^{n-1}}$ if $t > 3/4$ and $\theta^n(t) = t^{2^n}$ if $t \le 3/4$. Hence $\sum_{n=1}^{\infty}\sqrt{\theta^n(t)} < +\infty$ for all t in $\mathbb{R}^+$. We note that $\theta((3/4)+) = 3/4$.

Example 8. Define $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ as $\theta(t) = t/(1+t)$ for all t in $\mathbb{R}^+$. Then θ is a strictly increasing continuous function on $\mathbb{R}^+$ with $\theta(t) < t$ for all t in $(0,\infty)$. We have $\theta(1/n) = 1/(n+1)$ for all $n = 1,2,3,\ldots$. Hence $\theta^n(1) = 1/(n+1)$ for all $n = 1,2,3,\ldots$. Hence $\sum_{n=1}^{\infty}\sqrt{\theta^n(1)}$ is divergent.

We need the following lemma of Naidu [3].

Lemma 9 (see [3]). *Let $\{y_n\}_{n=0}^{\infty}$ be a sequence in (X,d). For $a \in X$, let $d_n(a) = d(y_n, y_{n+1},a)$. Suppose that $d_n(y_m) = 0$ for any nonnegative integers m, n with $n > m$. Then $d(y_i,y_j,y_k) = 0$ for all nonnegative integers i, j, k.*

Proposition 10. *Suppose that*

$$\begin{aligned} d^2(fx,gy,a) \le \varphi\big(&Kd(fx,Ty,a)d(Sx,gy,a) \\ &+\max\{d^2(Sx,Ty,a),d^2(Sx,fx,a),d^2(Ty,gy,a)\}\big) \\ &+\Psi\big(d(fx,Ty,a)d(Sx,gy,a)\big) \end{aligned} \tag{2}$$

for all x, y, a in X. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in X such that

$$fx_{2n} = Tx_{2n+1}(= y_{2n},\ say), \quad gx_{2n+1} = Sx_{2n+2}(= y_{2n+1},\ say) \quad (n = 0, 1, 2, \dots). \tag{3}$$

Then $\{y_n\}_{n=0}^{\infty}$ is Cauchy.

Proof. Let $d_n(a) = d(y_n, y_{n+1}, a)$. By taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in inequality (2) we obtain

$$d_{2n+1}^2(a) \le \varphi(\max\{d_{2n}^2(a), d_{2n+1}^2(a)\}). \tag{4}$$

By taking $x = x_{2n}$ and $y = x_{2n+1}$ in inequality (2) we obtain

$$d_{2n}^2(a) \le \varphi(\max\{d_{2n-1}^2(a), d_{2n}^2(a)\}). \tag{5}$$

From the above two inequalities we have

$$d_{n+1}^2(a) \le \varphi(\max\{d_n^2(a), d_{n+1}^2(a)\}) \quad (n = 0, 1, 2, \dots). \tag{6}$$

Since φ is nonnegative and $\varphi(t) < t$ for all t in $(0, \infty)$, from the above inequality we have

$$d_{n+1}^2(a) \le \varphi(d_n^2(a)) \quad (n = 0, 1, 2, \dots). \tag{7}$$

By repeatedly using inequality (7) and the monotonic increasing nature of φ we obtain

$$d_n^2(a) \le \varphi^n(d_0^2(a)) \quad (n = 0, 1, 2, \dots). \tag{8}$$

From inequality (7) we see that $d_{n+1}(a) = 0$ if $d_n(a) = 0$. Since $d_m(y_m) = 0$ for every nonnegative integer m, it follows that $d_n(y_m) = 0$ for any nonnegative integers m, n with $n > m$. Hence from Lemma 9 we have $d(y_i, y_j, y_k) = 0$ for all nonnegative integers i, j, k. Hence for any nonnegative integers m and n with $n < m$, by repeatedly using the triangle type inequality for 2-metrics, we obtain

$$d(y_n, y_m, a) \le \sum_{k=n}^{m-1} d_k(a). \tag{9}$$

Hence from inequality (8) we have

$$d(y_n, y_m, a) \le \sum_{k=n}^{m-1} \sqrt{\varphi^k(t_0)}, \tag{10}$$

where $t_0 = d_0^2(a)$. Since $\sum_{k=1}^{\infty} \sqrt{\varphi^k(t)} < +\infty$ for all t in $\mathbb{R}^+$, $\sum_{k=n}^{m-1} \varphi^k(t_0)$ tends to zero as both m and n tend to $+\infty$. Hence $d(y_n, y_m, a)$ tends to zero as both m and n tend to $+\infty$. Since this is true for any a in X, it follows that $\{y_n\}$ is Cauchy. □

Theorem 11. *Suppose that Ψ is right continuous at zero and*

$$\begin{aligned} d^2(fx, gy, a) \le \varphi(\max\{&d^2(x, y, a), d^2(x, fx, a), \\ &d^2(y, gy, a), Kd(fx, y, a)d(x, gy, a)\}) \\ &+ \Psi(d(fx, y, a)d(x, gy, a)) \end{aligned} \tag{11}$$

for all x,y,a in X. For any x_0 in X, let $\{x_n\}_{n=1}^{\infty}$ be defined iteratively as

$$x_{2n+1} = fx_{2n}, \quad x_{2n+2} = gx_{2n+1} \quad (n = 0,1,2,\dots). \tag{12}$$

Then $\{x_n\}$ is Cauchy. If $\{x_n\}$ converges to an element z of X, then z is a common fixed point of f and g. Further the fixed point sets of f and g are the same.

Proof. By taking $S = T = I$, the identity map on X, in Proposition 10, we can conclude that $\{x_n\}$ is a Cauchy sequence in X. Suppose that it converges to an element z of X. By taking $x = x_{2n}$ and $y = z$ in inequality (11) we obtain

$$\begin{aligned} d^2(x_{2n+1},gz,a) \le \varphi(\max\{&d^2(x_{2n},z,a),d^2(x_{2n},x_{2n+1},a), \\ &d^2(z,gz,a),Kd(x_{2n+1},z,a)d(x_{2n},gz,a)\}) \\ &+\Psi(d(x_{2n+1},z,a)d(x_{2n},gz,a)). \end{aligned} \tag{13}$$

The limit of the first term on the right-hand side of inequality (13) as n tends to $+\infty$ is $\varphi(d^2(z,gz,a))$ if $d(z,gz,a)$ is positive. Otherwise, it is $\varphi(0)$ or $\varphi(0+)$. Since $\varphi(0+) = \varphi(0) = 0$, in either case it can be written as $\varphi(d^2(z,gz,a))$. Since $\Psi(0+) = \Psi(0) = 0$, by taking limits on both sides of the above inequality as n tends to $+\infty$ we obtain

$$d^2(z,gz,a) \le \varphi(d^2(z,gz,a)). \tag{14}$$

Since $\varphi(t) < t$ for all t in $(0,\infty)$, we have $d^2(z,gz,a) = 0$. Since this is true for all a in X, $gz = z$. Similarly it can be shown that $fz = z$. If x is a fixed point of f, then by taking $y = x$ in inequality (11) we obtain $d^2(x,gx,a) \le \varphi(d^2(x,gx,a))$. Hence $gx = x$. Similarly it can be shown that any fixed point of g is also a fixed point of f. Hence f and g have the same fixed point sets. □

Remark 12. The hypothesis of Theorem 11 does not ensure the uniqueness of the common fixed point for f and g. This can be seen by taking f and g as identity maps on X, $K = 2$, $\varphi(t) = (1/2)t$ and $\Psi(t) = 0$ for all t in $\mathbb{R}^+$. We can also take $K = 0$ and $\varphi(t) = \Psi(t) = (1/2)t$ for all t in $\mathbb{R}^+$ or $\varphi(t) = 0$ and $\Psi(t) = t$ for all t in $\mathbb{R}^+$. Theorem 11 is an improvement over the existence part of Theorem 3 of Naidu [3] in which the first Ψ occurring in the governing inequality is to be read as φ. Proposition 10 is also an improvement over that of Naidu [3].

Corollary 13. *Suppose that (X,d) is complete and*

$$d^2(fx,fy,a) \le \alpha d(x,fx,a)d(y,fy,a) + \beta d(x,fy,a)d(fx,y,a) \tag{15}$$

for all x,y in X for some nonnegative constants α, β with $\alpha < 1$. Then f has a fixed point in X. If further $\beta < 1$, then f has a unique fixed point in X.

Proof. The existence part of the corollary follows from Theorem 11 by taking $g = f$, $K = 0$, $\varphi(t) = \alpha t$, and $\Psi(t) = \beta t$ for all t in $\mathbb{R}^+$. The rest of it is evident. □

Remark 14. Corollary 13 is the 2-metric space version of Theorem 1.

A perusal of the proof of Theorem 11 leads to the following variant.

Theorem 15. *Suppose that $\varphi(t+) < t$ for all t in $(0,\infty)$, Ψ is right continuous at zero and*

$$d^2(fx,gy,a) \le \varphi\big(Kd(fx,y,a)d(x,gy,a) + \max\{d^2(x,y,a),d^2(x,fx,a),d^2(y,gy,a)\}\big) + \Psi\big(d(fx,y,a)d(x,gy,a)\big) \tag{16}$$

for all x,y,a in X. For any x_0 in X, let $\{x_n\}_{n=1}^{\infty}$ be defined iteratively as in Theorem 11. Then $\{x_n\}$ is Cauchy. If $\{x_n\}$ converges to an element z of X, then z is a common fixed point of f and g. Further the fixed point sets of f and g are the same.

Remark 16. The hypothesis of Theorem 15 does not ensure the uniqueness of a common fixed point for f and g. This can be seen by taking f and g as identity maps on X, $K = 1$, $\varphi(t) = (1/2)t$, and $\Psi(t) = 0$ for all t in $\mathbb{R}^+$.

The concept of weak continuity of a 2-metric and that of weak commutativity for a pair of self-maps on a 2-metric space were introduced by Naidu and Prasad [4]. The notion of compatibility for a pair of self-maps on a metric space and that of weak compatibility for a pair of self-maps on an arbitrary set can be found in Jeong and Rhoades [2]. We state them below for the sake of completeness.

Definition 17 (see [4])**.** We say that d is weakly continuous at $z \in X$ if every convergent sequence in X with limit z is Cauchy.

Definition 18 (see [4])**.** A pair (f_1,f_2) of self-maps on (X,d) is said to be a weakly commuting pair (w.c.p.) if $d(f_1f_2x,f_2f_1x,a) \le d(f_2x,f_1x,a)$ for all x,y,a in X.

Definition 19 (see [2])**.** A pair (f_1,f_2) of self-maps on (M,ρ) is said to be a compatible pair (co.p.) if $\{\rho(f_1f_2x_n,f_2f_1x_n)\}$ converges to zero whenever $\{x_n\}$ is a sequence in M such that $\{f_1x_n\}$ and $\{f_2x_n\}$ are convergent in M and have the same limit.

Definition 20 (see [2])**.** A pair (f_1,f_2) of self-maps on an arbitrary set E is said to be a weakly compatible pair (w.co.p.) if $f_1f_2x = f_2f_1x$ whenever $x \in E$ is such that $f_1x = f_2x$.

In analogy with Definition 19 we introduce the concept of compatibility for a pair of self-maps on a 2-metric space.

Definition 21. A pair (f_1,f_2) of self-maps on (X,d) is called a compatible pair (co.p.) if $\{d(f_1f_2x_n,f_2f_1x_n,a)\}$ converges to zero for each a in X whenever $\{x_n\}$ is a sequence in X such that $\{f_1x_n\}$ and $\{f_2x_n\}$ are convergent sequences in X having the same limit and $\{d(f_2x_n,f_1x_n,a)\}$ converges to zero for each a in X.

Remark 22. The notion of asymptotic weak commutativity for a pair of self-maps on a 2-metric space introduced by Naidu [3] is slightly more stringent than the notion of compatibility introduced here. In 2-metric spaces, weak commutativity implies compatibility. But the converse is false. The following example illustrates it.

Example 23. Define d on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ as $d(x,y,z) = \min\{|x-y|,|y-z|,|z-x|\}$. Then d is a 2-metric on $\mathbb{R}^+$. Define f_1, f_2 from $\mathbb{R}^+$ to $\mathbb{R}^+$ as $f_1x = x/(1+x)$ and $f_2x = 2x$ for all x in $\mathbb{R}^+$. Let $\{x_n\}$ be a sequence in $\mathbb{R}^+$. Then $\{d(f_2x_n,f_1x_n,a)\}$

converges to zero for each a in $\mathbb{R}^+$ if and only if $\{x_n\}$ converges to zero in $\mathbb{R}^+$ in the usual sense. We have

$$\begin{aligned} d(f_1f_2x_n, f_2f_1x_n, a) &= \left(\frac{2x_n}{1+2x_n}, \frac{2x_n}{1+x_n}, a\right) \\ &\leq 2x_n\left(\frac{1}{1+x_n} - \frac{1}{1+2x_n}\right) \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty \end{aligned} \tag{17}$$

when $\{x_n\}$ converges to zero in $\mathbb{R}^+$ in the usual sense. Hence (f_1, f_2) is a co.p. For any positive real number x, while $d(f_1f_2x, f_2f_1x, 2x)$ is positive $d(f_2x, f_1x, 2x)$ is zero. Hence f_1 and f_2 do not commute weakly.

Theorem 24. *Suppose that* Ψ *is monotonically increasing on* $\mathbb{R}^+$, $\Psi(t+) < t$ *for all* t *in* $(0,\infty)$,

$$\begin{aligned} d^2(fx, gy, a) \leq \varphi(&\max\{d^2(Sx, fx, a), d^2(Ty, gy, a)\}) \\ &+ \Psi(d(fx, Ty, a)d(Sx, gy, a)) \end{aligned} \tag{18}$$

for all x, y, a *in* X *and that there are sequences* $\{x_n\}_{n=0}^{\infty}$ *and* $\{y_n\}_{n=0}^{\infty}$ *as stated in Proposition 10. Then* $\{y_n\}$ *is Cauchy. Suppose that it converges to an element* z *of* X. *Then the following statements are true.*

(I) *Neither the pair of maps* f *and* S *nor the pair of maps* g *and* T *can have a common fixed point other than* z. *If* $Sz = z$, *then* $fz = z$. *If* $Tz = z$, *then* $gz = z$.

(II) *If* $Sz = Tz$, *then* z *is a common fixed point of* f *and* S *if and only if it is a common fixed point of* g *and* T.

(III) *The point* z *is a unique common fixed point of* f *and* S *if one of the following five groups of conditions is true.*

- (i) (f,S) *is a w.co.p. and* $z \in S(X)$.
- (ii) (f,S) *is a co.p., and* f *and* S *are continuous at* z.
- (iii) (f,S) *is a co.p.,* S *is continuous at* z *and* d *is weakly continuous at* Sz.
- (iv) (f,S) *is a w.co.p. and, for some positive integer* k, $fS^k = S^kf$, S^k *is continuous at* z *and* d *is weakly continuous at* S^kz.
- (v) (f,S) *is a co.p. and, for some positive integer* k, S^k *is continuous at* z *and commutes with each of the maps* f, g, *and* T.

(IV) *The point* z *is a fixed point of* f *if one of the following two groups of conditions is true.*

- (i) (f,S) *is a co.p.,* f *is continuous at* z *and* d *is weakly continuous at* fz.
- (ii) f *is continuous at* z *and commutes with each of the maps* g, S, *and* T.

(V) *Statements* (III) and (IV) with f, g, S, and T replaced by g, f, T, and S, respectively.

Proof. That $\{y_n\}$ is Cauchy follows from Proposition 10. Suppose that it converges to an element z of X. By taking $y = x_{2n+1}$ in inequality (18) we obtain

$$\begin{aligned} d^2(fx, y_{2n+1}, a) \leq \varphi(&\max\{d^2(Sx, fx, a), d^2(y_{2n}, y_{2n+1}, a)\}) \\ &+ \Psi(d(fx, y_{2n}, a)d(Sx, y_{2n+1}, a)). \end{aligned} \tag{19}$$

By taking limits on both sides of inequality (19) as $n \to +\infty$ and using the facts that

$\{y_n\}$ is a Cauchy sequence converging to z, $\varphi(0) = \varphi(0+) = 0$ and Ψ is monotonically increasing on $\mathbb{R}^+$ we obtain

$$d^2(fx,z,a) \le \varphi(d^2(Sx,fx,a)) + \Psi(d(fx,z,a)d(Sx,z,a)^+) \tag{20}$$

for all a in X.

We now prove the following statements.

(1) If $fx = Sx$ for some $x \in X$, then $fx = Sx = z$.

(2) If $Sx = z$ for some $x \in X$, then $fx = z$.

Proof of (1). Suppose that $fx = Sx$ for some $x \in X$. Then, since $\varphi(0) = 0$, from inequality (20) we have $d^2(fx,z,a) \le \Psi(d^2(fx,z,a)^+)$. Since $\Psi(t+) < t$ for all t in $(0,\infty)$, we have $d^2(fx,z,a) = 0$. Since this is true for all a in X, we have $fx = z$.

Proof of (2). Suppose that $Sx = z$ for some $x \in X$. Then, since $\Psi(0+) = 0$, from inequality (20) we have $d^2(fx,z,a) \le \varphi(d^2(z,fx,a))$. Since $\varphi(t) < t$ for all t in $(0,\infty)$, we have $d^2(fx,z,a) = 0$. Since this is true for all a in X, we have $fx = z$.

Since inequality (18) remains unaffected if we interchange f, g, S, T with g, f, T, S, respectively, in analogy with statements (1) and (2) we have the following statements.

(3) If $gx = Tx$ for some $x \in X$, then $gx = Tx = z$.

(4) If $Tx = z$ for some $x \in X$, then $gx = z$.

Statement (I) is evident from statements (1), (2), (3), and (4). Statement (II) is evident from statement (I). We now prove the following statement (5).

(5) If S is continuous at z and (f,S) is a co.p., then $\{fSx_{2n}\}$ converges to Sz.

Suppose that S is continuous at z and (f,S) is a co.p. Since $d(Sx_{2n},fx_{2n},a) = d(y_{2n-1},y_{2n},a) \to 0$ as $n \to +\infty$, $\{y_n\}$ converges to z and (f,S) is a co.p., $d(fSx_{2n},Sfx_{2n},a) \to 0$ as $n \to +\infty$. Since S is continuous at z and $\{fx_{2n}\}$ converges to z, $\{Sfx_{2n}\}$ converges to Sz. We have

$$\begin{aligned} d(fSx_{2n},Sz,a) &\le d(fSx_{2n},Sfx_{2n},a) \\ &\quad + d(Sfx_{2n},Sz,a) + d(fSx_{2n},Sfx_{2n},Sz) \longrightarrow 0 \end{aligned} \tag{21}$$

as $n \to +\infty$. Hence $\{fSx_{2n}\}$ converges to Sz.

We now prove statement (III).

(i) Suppose that $z \in S(X)$. Then there exists an $x \in X$ such that $z = Sx$. From statement (2) it follows that $fx = z$. Suppose that (f,S) is a w.co.p. Then, since $Sx = fx$, we have $fSx = Sfx$, that is, $fz = Sz$. Hence from statement (1) we have $fz = Sz = z$.

(ii) Suppose that f and S are continuous at z and (f,S) is a co.p. From statement (5) it follows that $\{fSx_{2n}\}$ converges to Sz. Since f is continuous at z and $\{Sx_{2n}\}$ converges to z, $\{fSx_{2n}\}$ converges to fz. Hence $fz = Sz$. Hence from statement (1) we have $fz = Sz = z$.

(iii) Suppose that d is weakly continuous at Sz, S is continuous at z and (f,S) is a co.p. From statement (5) it follows that $\{fSx_{2n}\}$ converges to Sz. Since S is continuous at z and $\{Sx_{2n}\}$ converges to z, $\{SSx_{2n}\}$ converges to Sz. Since d is weakly continuous at Sz and both $\{fSx_{2n}\}$ and $\{SSx_{2n}\}$ converge to Sz, it follows that

$\{d(fSx_{2n},SSx_{2n},a)\}$ converges to zero. By taking $x=Sx_{2n}$ in inequality (20) we obtain

$$\begin{aligned} d^2(fSx_{2n},z,a) &\le \varphi(d^2(SSx_{2n},fSx_{2n},a)) \\ &\quad + \Psi(d(fSx_{2n},z,a)d(SSx_{2n},z,a)^+). \end{aligned} \tag{22}$$

By taking limits on both sides of (22) as $n \to +\infty$, we obtain $d^2(Sz,z,a) \le \Psi(d^2(Sz,z,a)^+)$. Since $\Psi(t+) < t$ for all t in $(0,\infty)$, we have $d^2(Sz,z,a) = 0$. Since this is true for all a in X, we have $Sz = z$. Hence from statement (I) we have $fz = z$.

We now establish the following statement (6), which is needed to complete the proof of statement (III).

(6) If (f,S) is a w.co.p. and, for some positive integer k, $fS^k = S^kf$, S^k is continuous at z, and $\{d(S^ky_{2n-1},S^ky_{2n},a)\}$ converges to zero for each a in X, then $fz = Sz = z$. Suppose that there is a positive integer k such that $fS^k = S^kf$, S^k is continuous at z and $\{d(S^ky_{2n-1},S^ky_{2n},a)\}$ converges to zero for each a in X. Since S^k is continuous at z and $\{y_n\}$ converges to z, $\{S^ky_n\}$ converges to S^kz. Since $fS^k = S^kf$, $fS^kx_{2n} = S^kfx_{2n} = S^ky_{2n}$. We have $SS^kx_{2n} = S^kSx_{2n} = S^ky_{2n-1}$. By taking $x = S^kx_{2n}$ in inequality (20), we obtain

$$\begin{aligned} d^2(S^ky_{2n},z,a) &\le \varphi(d^2(S^ky_{2n-1},S^ky_{2n},a)) \\ &\quad + \Psi(d(S^ky_{2n},z,a)d(S^ky_{2n-1},z,a)^+). \end{aligned} \tag{23}$$

By taking limits on both sides of (23) as $n \to +\infty$, we obtain $d^2(S^kz,z,a) \le \Psi(d^2(S^kz,z,a)^+)$. Since $\Psi(t+) < t$ for all t in $(0,\infty)$, we have $d^2(S^kz,z,a) = 0$. Since this is true for all a in X, we have $S^kz = z$. Hence $z \in S(X)$.

Hence, if (f,S) is a w.co.p., then conditions (i) of statement (III) are fulfilled so that from what we already proved we have $fz = Sz = z$.

We now resume the proof of statement (III).

(iv) Suppose that S^k is continuous at z for some positive integer k. Then $\{S^ky_n\}$ converges to S^kz. Suppose now that d is weakly continuous at S^kz. Then $\{d(S^k y_{2n-1},S^ky_{2n},a)\}$ converges to zero for each a in X. Hence, if (f,S) is a w.co.p. and $fS^k = S^kf$, then from statement (6) we have $fz = Sz = z$.

(v) Suppose that S^k commutes with each of the maps f, g, and T for some positive integer k. Then from equation (3) we have $f(S^kx_{2n}) = T(S^kx_{2n+1}) = S^ky_{2n}$ and $g(S^kx_{2n+1}) = S(S^kx_{2n+2}) = S^ky_{2n+1}$ for all $n = 0,1,2,\ldots$. Hence from Proposition 10, it follows that $\{S^ky_n\}$ is Cauchy. In particular, $\{d(S^ky_{2n-1},S^ky_{2n},a)\}$ converges to zero for each a in X. Hence from statement (6) it follows that $fz = Sz = z$ if conditions (v) of statement (III) are fulfilled.

The proof of the following statement is similar to that of statement (5).

(7) If f is continuous at z and (f,S) is a co.p., then $\{Sfx_{2n}\}$ converges to fz.

We now prove statement (IV).

(i) Suppose that d is weakly continuous at fz, f is continuous at z and (f,S) is a co.p. From statement (7) it follows that $\{Sfx_{2n}\}$ converges to fz. Since f is continuous at z and $\{fx_{2n}\}$ converges to z, $\{ffx_{2n}\}$ converges to fz. Since d is weakly continuous at fz and both $\{Sfx_{2n}\}$ and $\{ffx_{2n}\}$ converge to fz, $\{d(Sfx_{2n},ffx_{2n},a)\}$ converges to zero. By taking $x = fx_{2n}$ in inequality (20) and then taking limits on both sides of the inequality as $n \to +\infty$, we obtain $d^2(fz,z,a) \le \Psi(d^2(fz,z,a)^+)$.

Since $\Psi(t+) < t$ for all t in $(0,\infty)$, we have $d^2(fz,z,a) = 0$. Since this is true for all a in X, we have $fz = z$.

We note that if $fS = Sf$, then $d(Sfx_{2n},ffx_{2n},a) = d(fSx_{2n},ffx_{2n},a) = d(fy_{2n-1},fy_{2n},a)$ and (f,S) is a co.p. Hence from a perusal of the above proof we have the following statement.

(8) If $fS = Sf$, f is continuous at z and $\{d(fy_{2n-1},fy_{2n},a)\}$ converges to zero for each a in X, then $fz = z$.

We now resume the proof of statement (IV).

(ii) Suppose that f commutes with each of the maps g, S, and T. Then from equations (3) we have

$$f(fx_{2n}) = T(fx_{2n+1}) = fy_{2n},$$
$$g(fx_{2n+1}) = S(fx_{2n+2}) = fy_{2n+1} \quad \forall n = 0,1,2,\ldots. \tag{24}$$

Hence from Proposition 10, it follows that $\{fy_n\}$ is Cauchy. Hence $\{d(fy_{2n-1},fy_{2n},a)\}$ converges to zero for each a in X. Hence from statement (8) it follows that $fz = z$ if conditions (ii) of statement (IV) are fulfilled.

Statement (V) follows from symmetry considerations. □

Remark 25. Theorem 24 is an improvement over Theorem 2 in [3].

Corollary 26. *Suppose that Ψ is monotonically increasing on $\mathbb{R}^+$, $\Psi(t+) < t$ for all t in $(0,\infty)$,*

$$d^2(fx,gy,a) \le \varphi(\max\{d^2(Sx,fx,a),d^2(Sy,gy,a)\}) + \Psi(d(fx,Sy,a)d(Sx,gy,a)) \tag{25}$$

for all x, y, a in X and that there are sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ in X such that

$$fx_{2n} = Sx_{2n+1} = y_{2n}, \quad gx_{2n+1} = Sx_{2n+2} = y_{2n+1} \quad (n = 0,1,2,\ldots). \tag{26}$$

Then $\{y_n\}$ is Cauchy. Suppose that it converges to an element z of X. Then z is a unique common fixed point of f, g, and S if one of the following groups of conditions is true.

(i) *$z \in S(X)$ and either (f,S) or (g,S) is a w.co.p.*

(ii) *S is continuous at z, and either f is continuous at z and (f,S) is a co.p. or g is continuous at z and (g,S) is a co.p.*

(iii) *S is continuous at z, d is weakly continuous at Sz and either (f,S) or (g,S) is a co.p.*

(iv) *S^k is continuous at z and d is weakly continuous at S^kz for some positive integer k, and S commutes with either f or g.*

(v) *S commutes with each of the maps f and g, and S^k is continuous at z for some positive integer k.*

Remark 27. In Theorem 24 if inequality (18) is replaced with the following more stringent inequality

$$d^2(fx,gy,a) \le \varphi(d(Sx,fx,a)d(Ty,gy,a)) + \Psi(d(fx,Ty,a)d(Sx,gy,a)), \tag{27}$$

then the weak continuity of d can be dropped from all those numbered statements in which it appears. A similar remark applies to Corollary 26 also.

We now state without proof the metric space versions of some of the results we obtained in 2-metric spaces. Hereafter, unless otherwise stated, f, g, S, T are self-maps on M.

Proposition 28. *Suppose that*

$$\begin{aligned}\rho^2(fx,gy) \le \varphi\big(K\rho(fx,Ty)\rho(Sx,gy) \\ +\max\{\rho^2(Sx,Ty),\rho^2(Sx,fx),\rho^2(Ty,gy)\}\big) \\ +\Psi\big(\rho(fx,Ty)\rho(Sx,gy)\big)\end{aligned} \tag{28}$$

for all x, y in M and that there are sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ in M satisfying equations (3). Then $\{y_n\}_{n=0}^{\infty}$ is Cauchy.

Remark 29. Proposition 28 fails if the condition $\sum_{n=1}^{\infty}\sqrt{\varphi^n(t)} < +\infty$ for all t in $\mathbb{R}^+$ is replaced by the condition $\varphi(t+) < t$ for all t in $(0,\infty)$. Example 30 illustrates this when $g = f$, $S = T = I$ (the identity map on M) and $\Psi(t) = t$ for all t in $\mathbb{R}^+$.

Example 30. Let $M = \{x_n : n = 1,2,3,\dots\}$, where $x_n = \sum_{k=1}^{n} 1/k$. Define $f : M \to M$ as $fx_n = x_{n+1}$ for all $n = 1,2,3,\dots$. Define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi(t) = t/(1+t)$ for all t in $\mathbb{R}^+$. Then φ is a strictly increasing continuous function on $\mathbb{R}^+$ with $\varphi(t) < t$ for all t in $(0,\infty)$ and

$$|fx-fy|^2 \le \varphi(|x-y|^2) + |fx-y|\,|x-fy| \tag{29}$$

for all x,y in M. Evidently for any x in M the sequence $\{f^n x\}$ diverges to $+\infty$ and hence is not Cauchy.

Theorem 31. *Suppose that Ψ is right continuous at zero and*

$$\begin{aligned}\rho^2(fx,gy) \le \varphi\big(\max\{\rho^2(x,y),\rho^2(x,fx),\rho^2(y,gy),K\rho(fx,y)\rho(x,gy)\}\big) \\ +\Psi\big(\rho(fx,y)\rho(x,gy)\big)\end{aligned} \tag{30}$$

for all x,y in M. For any x_0 in M, let $\{x_n\}_{n=1}^{\infty}$ be defined iteratively as in Theorem 11. Then $\{x_n\}$ is Cauchy. If $\{x_n\}$ converges to an element z of M, then z is a common fixed point of f and g. Further the fixed point sets of f and g are the same.

Theorem 32. *Suppose that $\varphi(t+) < t$ for all t in $(0,\infty)$, Ψ is right continuous at zero and*

$$\begin{aligned}\rho^2(fx,gy) \le \varphi\big(K\rho(fx,y)\rho(x,gy) \\ +\max\{\rho^2(x,y),\rho^2(x,fx),\rho^2(y,gy)\}\big) \\ +\Psi\big(\rho(fx,y)\rho(x,gy)\big)\end{aligned} \tag{31}$$

for all x,y in M. For any x_0 in M, let $\{x_n\}_{n=1}^{\infty}$ be defined iteratively as in Theorem 11. Then $\{x_n\}$ is Cauchy. If $\{x_n\}$ converges to an element z of M, then z is a common fixed point of f and g. Further the fixed point sets of f and g are the same.

Remark 33. In Theorem 32 the conclusion that z is a common fixed point of f and g fails in the absence of the condition $\varphi(t+) < t$ for all t in $(0,\infty)$ even if (M,ρ) is complete, $f = g$, $K = 1$ and Ψ is identically zero on $\mathbb{R}^+$. Examples 34 and 35 illustrate this. While in Example 34 the function f has no fixed point, in Example 35 it has.

Example 34. Let $M = \{1/2^n \mid n = 0,1,2,\dots\} \cup \{0\}$. Then M is a complete metric space under the metric induced by the modulus function. Define $f : M \to M$ as $fx = (1/2)x$ if $x \neq 0$ and $f0 = 1$. Define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi(t) = 1$ if $t > 1$ and $\varphi(t) = (1/4)t$ if $t \le 1$. Then φ is monotonically increasing on $\mathbb{R}^+$, $\sum_{n=1}^{\infty}\sqrt{\varphi^n(t)} < +\infty$ for all t in $\mathbb{R}^+$, $\varphi(1+) = 1$ and

$$|fx - fy|^2 \le \varphi\big(|fx - y|\,|x - fy| + \max\{|x-y|^2, |x-fx|^2, |y-fy|^2\}\big) \tag{32}$$

for all x, y in M. We note that for any x_0 in M the sequence $\{f^n x_0\}$ converges to zero. But 0 is not a fixed point of f. In fact, f has no fixed point.

Example 35. Let M be as in Example 34. Define $f : M \to M$ as $fx = (1/2)x$ if $x \notin \{0,1\}$ and $f0 = f1 = 1$. Define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi(t) = 1$ if $t > 1$ and $\varphi(t) = (9/10)t$ if $t \le 1$. Then φ is monotonically increasing on $\mathbb{R}^+$, $\sum_{n=1}^{\infty}\sqrt{\varphi^n(t)} < +\infty$ for all t in $\mathbb{R}^+$, $\varphi(1+) = 1$ and inequality (32) is satisfied for all x, y in M. We note that for any x_0 in $M\backslash\{0,1\}$ the sequence $\{f^n x_0\}$ converges to zero. But 0 is not a fixed point of f.

Remark 36. A pair (f_1, f_2) of self-maps on (M,ρ) is a w*.c.p. if $\rho(f_1f_2x, f_2f_1x) \le \gamma\rho(f_2x, f_1x)$ for all x in M for some nonnegative real number γ and a w.c.p. (weakly commuting pair) if $\rho(f_1f_2x, f_2f_1x) \le \rho(f_2x, f_1x)$ for all x in M. (The notion of weak commutativity for a pair of self-maps on a metric space was introduced by Sessa [5].) Clearly a w.c.p. is a w*.c.p. and a w*.c.p. is a co.p. But the converse is false in either case. Examples 37 and 38 illustrate this.

Example 37. Define f_1, f_2 from $\mathbb{R}$ to $\mathbb{R}$ as $f_1x = x^2$ and $f_2x = 2x - 1$ for all x in $\mathbb{R}$. Then $|f_1f_2x - f_2f_1x| = 2(x-1)^2 = 2|f_1x - f_2x|$ for all x in $\mathbb{R}$. Hence (f_1, f_2) is a w*.c.p. but not a w.c.p.

Example 38. Define f_1, f_2 from $\mathbb{R}$ to $\mathbb{R}$ as $f_1x = x^2$ and $f_2x = -x^2$ for all x in $\mathbb{R}$. Then $|f_1f_2x - f_2f_1x| = 2x^4$ and $|f_1x - f_2x| = 2x^2$ for all x in $\mathbb{R}$. Clearly there is no nonnegative real number γ such that $2x^4 \le \gamma(2x^2)$ for all x in $\mathbb{R}$. Hence (f_1, f_2) is not a w*.c.p. Clearly it is a co.p.

Theorem 39. *Suppose that Ψ is monotonically increasing on $\mathbb{R}^+$, $\Psi(t+) < t$ for all t in $(0,\infty)$,*

$$\rho^2(fx, gy) \le \varphi\big(\max\{\rho^2(Sx, fx), \rho^2(Ty, gy)\}\big) + \Psi\big(\rho(fx, Ty)\rho(Sx, gy)\big) \tag{33}$$

for all x, y in M and that there are sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ in M satisfying equations (3). Then $\{y_n\}_{n=0}^{\infty}$ is Cauchy. Suppose that it converges to an element z of M. Then the following statements are true.

(1) *Neither the pair of maps f and S nor the pair of maps g and T can have a common fixed point other than z. If $Sz = z$, then $fz = z$. If $Tz = z$, then $gz = z$.*

(2) *If* $Sz = Tz$, *then* z *is a common fixed point of* f *and* S *if and only if it is a common fixed point of* g *and* T.

(3) *If* (f,S) *is a w.co.p. and* $z \in S(X)$, *then* $fz = Sz = z$.

(4) *If* (f,S) *is a co.p. and* S *is continuous at* z, *then* $fz = Sz = z$.

(5) *If* (f,S) *is a w.co.p. and, for some positive integer* k, $fS^k = S^k f$ *and* S^k *is continuous at* z, *then* $fz = Sz = z$.

(6) *If* f *is continuous at* z *and* (f,S) *is a co.p., then* $fz = z$.

(7) *Statements (3), (4), (5), and (6) with* f *and* S *replaced by* g *and* T, *respectively.*

Finally we conclude the paper with the following open problem.

Open problem. Does Theorem 15 remain valid if the condition $\varphi(t+) < t$ for all t in $(0,\infty)$ is deleted from the hypothesis?

Note 40. The results in Naidu and Prasad [4] remain valid if the weak commutativity condition in them is replaced with compatibility condition as introduced in Definition 21.

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