

TWISTED FORMS OF FINITE ÉTALE EXTENSIONS AND SEPARABLE POLYNOMIALS

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ABSTRACT. Examples of twisted forms of finite étale extensions and separable polynomials are calculated using Mayer-Vietoris sequences for non-abelian cohomology.

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1. Introduction. Let S be a finite étale extension of a commutative Noetherian ring R (a finitely generated projective separable extension of R). A twisted form of S (in the Zariski topology) is a finite étale extension T of R with $R_P \otimes S \cong R_P \otimes T$ as R_P -algebras for each prime ideal P of R . In this case S is locally isomorphic to T . If $Q \subset P$ are prime ideals of R and $R_P \otimes S \cong R_P \otimes T$, then $R_Q \otimes S \cong R_Q \otimes T$ so prime can be replaced by maximal in the definition of the twisted form. In this paper, we study the set of isomorphism classes of twisted forms of S . We especially concentrate on the case where $S \cong R[t]/(p(t))$, where $p(t)$ is a separable polynomial in $R[t]$. Throughout this paper, R denotes a commutative Noetherian ring.

We first observe the well-known facts that if R is an integrally closed domain, then there are no twisted forms of S and in general the twisted forms of S are in bijective correspondence with $H^1(X, \text{Aut}(\mathcal{S}))$, where $\text{Aut}_R(\mathcal{S})$ is the sheaf of automorphisms on $X = \text{Spec}(R)$ associated to $\text{Aut}_R(S)$. We check that $H^1(X, \text{Aut}(\mathcal{S}))$ is unchanged modulo a nilpotent ideal.

With some hypotheses on the sheaf $\text{Aut}_R(\mathcal{S})$, when R is a one-dimensional domain or if R is a reduced one-dimensional ring with connected spectrum, $H^1(X, \text{Aut}(\mathcal{S}))$ fits into a Mayer-Vietoris sequence which makes computations possible. These computations are the point of this article. For infinitely many prime numbers p, q we give a class of examples of integral domains R and separable polynomials $t^p - q \in R[t]$ with the cardinality of the set of isomorphism classes of twisted forms of $S \cong R[t]/(p(t))$ equal to $(p-1)!$. When $p = 3$ these twisted forms T are isomorphic to algebras $T \cong \bigoplus_{j=0}^2 I^j t^j$ with $t^3 = q$ and I is a fractional ideal with $I^3 = R$. We give an example of twisted forms that do not have this structure. We also give one-dimensional rings over which finite étale extensions may not have either a primitive element nor a normal basis but which are twisted forms of extensions which do. We also give a separable polynomial which is irreducible over R but factors into linear factors at each localization R_P of R and modulo each maximal ideal of R .

2. Mayer-Vietoris sequences and examples. Let R be a commutative Noetherian ring and S a finite étale R -algebra. Let $X = \text{Spec}(R)$ be the space of prime ideals of R

with the Zariski topology and \mathbb{O}_X the associated sheaf of rings on X , \mathcal{S} the sheaf of \mathbb{O}_X -algebras associated to S (see [5, pages 70 and 130]). For each open set $U \subset X$ associate to U the group of R -algebra automorphisms $\text{Aut}_{\mathbb{O}(U)}(\mathcal{S}(U))$, and if $V \subset U$ associate the natural restriction $\text{Aut}_{\mathbb{O}(U)}(\mathcal{S}(U)) \rightarrow \text{Aut}_{\mathbb{O}(V)}(\mathcal{S}(V))$. We begin by recording for the readers convenience that $\text{Aut}_R(\mathcal{S})$ forms a sheaf on X and give some of its properties (see also [3]).

LEMMA 2.1. *Let S be a finite étale R -algebra.*

(a) *If U is an open set in X and $\{V_i\}$ is an open cover of U , and if $\sigma \in \text{Aut}_{\mathbb{O}(U)}(\mathcal{S}(U))$ satisfies $1 \otimes \sigma = 1$ in $\text{Aut}_{\mathbb{O}(V_i)}(\mathcal{S}(V_i))$ for all i , then $\sigma = 1$ in $\text{Aut}_{\mathbb{O}(U)}(\mathcal{S}(U))$.*

(b) *If U is an open set in X and $\{V_i\}$ is an open cover of U , and if $\sigma_i \in \text{Aut}_{\mathbb{O}(V_i)}(\mathcal{S}(V_i))$ with restrictions $\sigma_i = \sigma_j$ in $\text{Aut}_{\mathbb{O}(V_i \cap V_j)}(\mathcal{S}(V_i \cap V_j))$, then there is an element $\sigma \in \text{Aut}_{\mathbb{O}(U)}(\mathcal{S}(U))$ whose restriction to $\text{Aut}_{\mathbb{O}(V_i)}(\mathcal{S}(V_i))$ is σ_i .*

LEMMA 2.2. *Let S, T be finite étale R -algebras and P a prime ideal in R . Let $\sigma : R_P \otimes S \rightarrow R_P \otimes T$ be an R_P -algebra homomorphism. Then there is an open set U in X with $P \in U$ and an $\mathbb{O}(U)$ -algebra homomorphism $\tau : \mathbb{O}(U) \otimes S \rightarrow \mathbb{O}(U) \otimes T$ such that $1 \otimes \tau = \sigma \in \text{Alg}_{R_P}(R_P \otimes S, R_P \otimes T)$. If σ is an isomorphism then U can be chosen so τ is an isomorphism.*

If S, T are finite étale R -algebras and $R_P \otimes S \cong R_P \otimes T$ for all prime ideals P of R we say that S and T are locally isomorphic or that T is a twisted form of S . If T is a twisted form of S then Lemma 2.2 implies there is an open cover $\mathcal{U} = \{U_i\}$ of X and isomorphisms $\sigma_i : \mathbb{O}(U_i) \otimes S \rightarrow \mathbb{O}(U_i) \otimes T$ for all i . Define an element $a \in Z^1(\mathcal{U}, \text{Aut}(\mathcal{S}))$ by assigning to the index pair i, j the automorphism $a(i, j) = \sigma_i^{-1} \sigma_j \in \text{Aut}_{\mathbb{O}(U_i \cap U_j)}(\mathbb{O}(U_i \cap U_j) \otimes S)$. Passing to the limit over all covers of X gives an injection from the set of isomorphism classes of twisted forms of S to $H^1(X, \text{Aut}(\mathcal{S}))$. By descent, (see [7, 2.2, page 110] or [8, page 19]), this assignment is onto so $H^1(X, \text{Aut}(\mathcal{S}))$ classifies the twisted forms of S . In the next result we point out that, as with the Brauer group, there are no nontrivial twisted forms in the geometrically irreducible case.

PROPOSITION 2.3. *If R is an integrally closed domain and S is a finite étale R -algebra, then $H^1(X, \text{Aut}(\mathcal{S})) = \{1\}$.*

PROOF. We can write $S = S_1 \oplus \dots \oplus S_k$, where each S_i has a connected spectrum. By [6, Theorem 4.3] or [8, Proposition 3.19, page 28], each S_i is an integrally closed domain. Let K be the quotient field of R . Then $K \otimes S = \bigoplus_{i=1}^k K \otimes S_i$, where each $K \otimes S_i$ is a finite-dimensional separable field extension of K , and S_i is the integral closure of R in $K \otimes S_i$. Let $\sigma \in \text{Aut}_K(K \otimes S)$, then since the image of an integral element is integral, $\sigma|_S \in \text{Aut}_R(S)$ and $\sigma = 1 \otimes \sigma|_S$. Therefore, the natural map $\text{Aut}_R(S) \rightarrow \text{Aut}_K(K \otimes S)$ is a bijection which implies that the sheaf $\text{Aut}(\mathcal{S})$ is constant. Therefore, $H^1(X, \text{Aut}(\mathcal{S})) = \{1\}$. □

COROLLARY 2.4. *Let R be an integrally closed domain and S, T finite étale R -algebras. If $R_P \otimes S \cong R_P \otimes T$ as R_P -algebras for each $P \in \text{Spec}(R)$, then $S \cong T$ as R -algebras.*

LEMMA 2.5. *Let S be a finite étale R -algebra, I an ideal in R , and $\rho : \text{Aut}_R(S) \rightarrow \text{Aut}_{R/I}(S/I)$ the natural map.*

- (a) If R has a connected spectrum then ρ is a one-to-one map.
- (b) If I is nilpotent then ρ is a bijection map.

PROOF. (a) Assume first that S is connected and Galois over R . Then $\text{Aut}_R(S) =$ Galois group of S over $R =$ the Galois group of S/IS over $R/I \subset \text{Aut}_{R/I}(S/IS)$ so ρ is a one-to-one map in this case. If S is connected but not necessarily Galois, imbed S in a connected Galois extension N of R (see [2, Theorem 3.2.9]). Every R -automorphism of S extends to an automorphism of N , and any two such extensions differ by an element of $H = \{\sigma \in \text{Aut}_R(N) \mid \sigma|_S = 1\}$ (see [2, Chapter 3]). By flatness, S/IS is a subalgebra of N/IN and H is the subgroup of the Galois group of N/IN over R/I fixing S/IS . If $\tau, \sigma \in \text{Aut}_R(S)$ with extensions $\bar{\tau}, \bar{\sigma}$ to N and with the natural image of $\tau = \sigma$ in $\text{Aut}_{R/I}(S/IS)$, then $\bar{\tau}^{-1}\bar{\sigma} \in H$ so $\tau = \sigma$ on S and ρ is a one-to-one map in this case.

If R is connected then $S = Se_1 \oplus \dots \oplus Se_m$ with $e_i e_j = e_i \delta_{i,j}$, Se_i connected for all i, j . Let $\sigma \in \text{Aut}_R(S)$ and assume σ induces the identity automorphism on S/IS . Then $\sigma(e_i) = e_i$ for all i since $e_i + IS \neq e_j + IS$ for any $i \neq j$. Therefore σ induces $(\sigma_1, \dots, \sigma_m) \in \times_i \text{Aut}_R(Se_i)$. Since σ is the identity on each of these summands, by the previous paragraph σ is the identity on S and ρ is a one-to-one map.

(b) We can write $R = R_1 \oplus \dots \oplus R_k$, where each R_i has a connected spectrum. Then there are the corresponding decompositions $S = S_1 \oplus \dots \oplus S_k$ and $I = I_1 \oplus \dots \oplus I_k$ with I_j a nilpotent ideal of R_j (in particular, no $I_j = R_j$ and $\text{Aut}_R(S) = \times_j \text{Aut}_{R_j}(S_j)$, $\text{Aut}_{R/I}(S/IS) = \times_j \text{Aut}_{R/I_j}(R_j \otimes S/I_j(R_j \otimes S))$). Thus we can assume R is connected.

If I is nilpotent, then IS is nilpotent and idempotents can be lifted modulo a nilpotent ideal, so R/I has a connected spectrum and if S is connected then S/IS is connected. Assume S is connected and Galois. Then $\text{Aut}_R(S) =$ the Galois group of S over $R =$ the Galois group S/IS over $R/I = \text{Aut}_{R/I}(S/IS)$ so ρ is bijective in this case. If S/R is Galois, then $S = Se_1 \oplus \dots \oplus Se_m$ with $e_i e_j = e_i \delta_{i,j}$, Se_i connected, $Se_i \cong Se_j$ for all i, j , and each Se_i Galois over R with Galois group of order $n = \text{rank}(Se_i)$. Thus $|\text{Aut}_R(S)| = m!n^m$. Since idempotents can be lifted modulo I , we get the same count for $|\text{Aut}_{R/I}(S/IS)|$, so by part (a), ρ is onto in this case.

If S/R is finite étale, there is a Galois extension N of R containing S constructed in the following way. Write $S = Se_1 \oplus \dots \oplus Se_m$ as above and let L be a connected Galois extension of R containing all the Se_i . Let $N = Le_1 \oplus \dots \oplus Le_m$. Let $\bar{\tau} \in \text{Aut}_{R/I}(S/IS)$. Then one can extend $\bar{\tau}$ to $\bar{y} \in \text{Aut}_{R/I}(N/IN)$ which corresponds, by the paragraph above, to $y \in \text{Aut}_R(N)$. Then $(y(S) + IS)/I = S/I$ so $y(S) \subset S$. Therefore $y|_S \in \text{Aut}_R(S)$ and $\rho(y|_S) = \bar{\tau}$. Thus ρ is a bijection in every case. □

COROLLARY 2.6. *Let $X = \text{Spec}(R)$, I the nil radical of R and $X_{\text{red}} = \text{Spec}(R/I)$. If S is a finite étale R -algebra, then $H^1(X, \text{Aut}(\mathcal{F}))$ and $H^1(X_{\text{red}}, \text{Aut}(\mathcal{F}/\mathcal{F}))$ are bijective with one another.*

PROOF. By Lemma 2.5, $X = X_{\text{red}}$ and $\text{Aut}(\mathcal{F}) = \text{Aut}(\mathcal{F}/\mathcal{F})$. □

EXAMPLE 2.7. (a) Let \mathbb{R} denote the real numbers and \mathbb{C} the complex numbers. Let $R = \mathbb{R} \oplus \mathbb{R}$ and $S = \mathbb{C} \oplus \mathbb{C}$. Let σ be complex conjugation. Then $(1, 1) = (1, \sigma)$ on the first summand but $(1, 1) \neq (1, \sigma)$ so the map $\text{Aut}_R(S) \rightarrow \text{Aut}_{R/I}(S/IS)$ is not always a one-to-one map.

(b) Let R be the localization of $\mathbb{C}[x]$ at (x) and let $p(t) = t^3 + (x + 1) \in R[t]$. Let

$S = R[t]/(p(t))$. Since $p(t)$ is irreducible, $\text{Aut}_R(S) = C_3$ (the cyclic group of order 3) but $R/(x) = \mathbb{C}$ and $S/(x)S = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ has \mathbb{C} -automorphism group S_3 (the symmetric group on three letters). It is not always the case that $\text{Aut}_R(S) \rightarrow \text{Aut}_{R/I}(S/IS)$ is onto.

MAYER-VIETORIS I. Let R be a one-dimensional integral domain with module finite integral closure \bar{R} and conductor $c = \{x \in R \mid \bar{R} \cdot x \subset R\}$. Let S be a finite étale R -algebra. Assume the following.

(a) If P is a maximal ideal in R containing c then there is only one maximal ideal Q in \bar{R} lying over P .

(b) If P is a maximal ideal in R containing c then the natural map $\text{Aut}_R(S) \rightarrow \text{Aut}_{R/P}(S/PS)$ is surjective.

(c) $\bar{R} \otimes S \cong \bar{R}^{(n)}$, where $n = \text{rank}_R(S)$.

Then there is an exact sequence of pointed sets

$$\begin{aligned} 1 \longrightarrow \text{Aut}_R(S) &\xrightarrow{\alpha} \text{Aut}_{\bar{R}}(\bar{R} \otimes S) \times \text{Aut}_{R/c}(R/c \otimes S) \\ &\xrightarrow{\beta} \text{Aut}_{\bar{R}/c}(\bar{R}/c \otimes S) \xrightarrow{\gamma} H^1(X, \text{Aut}(\mathcal{F})) \longrightarrow 1. \end{aligned} \tag{2.1}$$

SKETCH OF THE PROOF. We define explicitly the maps in the sequence. Checking exactness at each term is then a straightforward computation.

The map α is given as $\alpha(\sigma) = (1 \otimes \sigma, 1 \otimes \sigma)$. The map β is given as $\beta(\tau, \rho) = (1 \otimes \tau)(1 \otimes \rho)^{-1}$.

Let P_1, \dots, P_k be the maximal ideals of R lying over c . Using hypothesis (a), let Q_1, \dots, Q_k be the maximal ideals in \bar{R} lying over c with $Q_i \cap R = P_i$ ($1 \leq i \leq k$). Then $c = \cap_i P_i^{f_i} = \cap_i Q_i^{e_i}$ so $R/c = \oplus R/P_i^{f_i}$ and $\bar{R}/c = \oplus \bar{R}/Q_i^{e_i}$. Moreover, $\text{Aut}_{R/c}(S/cS) = \times_i \text{Aut}_{R/P_i^{f_i}}(R/P_i^{f_i} \otimes S)$ and $\text{Aut}_{\bar{R}/c}(\bar{R}/c \otimes S) = \times_i \text{Aut}_{\bar{R}/Q_i^{e_i}}(\bar{R}/Q_i^{e_i} \otimes S)$. If $(\dots, \bar{\sigma}_i, \dots) \in \text{Aut}_{\bar{R}/c}(\bar{R}/c \otimes S)$ then by hypothesis (c) and [Lemma 2.5](#), $\text{Aut}_{\bar{R}}(\bar{R} \otimes S) = S_n = \text{Aut}_{\bar{R}/Q_i^{e_i}}(\bar{R}/Q_i^{e_i} \otimes S)$ so there exists $\sigma_i \in \text{Aut}_{\bar{R}}(\bar{R} \otimes S)$ with σ_i a lift of $\bar{\sigma}_i$. Let $\mathcal{U} = \{U_j\}$ be a cover of $X = \text{Spec}(R)$ with $P_i \in U_j$ if and only if $i = j$. Assign to U_j the identity automorphism if no P_i is in U_j . Since $U_i \cap U_j$ does not contain any points lying over c , $\text{Aut}_{\mathcal{O}(U_i \cap U_j)}(\mathcal{F}(U_i \cap U_j)) = S_n$ and therefore contains the element $a(i, j) = \sigma_i^{-1} \sigma_j$. It is now easy to check $a \in Z^1(\mathcal{U}, \text{Aut}(\mathcal{F}))$ is a 1-cocycle and a different choice of cover gives an equivalent cocycle modulo coboundaries, so γ is defined by $\gamma(\dots, \bar{\sigma}_i, \dots) = |a| \in H^1(X, \text{Aut}(\mathcal{F}))$.

MAYER-VIETORIS II. Let R be a reduced ring with $X = \text{Spec}(R)$ connected. Let I_1, \dots, I_q be the set of minimal prime ideals of R and $\bar{R} = \oplus_{j=1}^q R/I_j$. Assume $\dim R/I_j = 1$ for all j . Identify R with its natural image in \bar{R} and let $c = \{x \in R \mid \bar{R} \cdot x \subset R\}$ be the conductor. Let $Y = \text{Spec}(\bar{R})$. Let S be a finite étale R -algebra.

(a) Assume for each maximal ideal Q in \bar{R} lying over c the natural map $\text{Aut}_R(S) \rightarrow \text{Aut}_{\bar{R}/Q}(\bar{R}/Q \otimes S)$ is a surjection.

Then there is an exact sequence of pointed sets

$$\begin{aligned} 1 \longrightarrow \text{Aut}_R(S) &\xrightarrow{\alpha} \text{Aut}_{\bar{R}}(\bar{R} \otimes S) \times \text{Aut}_{R/c}(R/c \otimes S) \\ &\xrightarrow{\beta} \text{Aut}_{\bar{R}/c}(\bar{R}/c \otimes S) \xrightarrow{\gamma} H^1(X, \text{Aut}(\mathcal{F})) \xrightarrow{\delta} H^1(Y, \text{Aut}(\bar{R} \otimes \mathcal{F})). \end{aligned} \tag{2.2}$$

SKETCH OF THE PROOF. As in Mayer-Vietoris I, we give the maps explicitly, then checking exactness is a straightforward computation. The map α is defined as $\alpha(\sigma) = (1 \otimes \sigma, 1 \otimes \sigma)$. The map β is $\beta(\tau, \rho) = (1 \otimes \tau)(1 \otimes \rho)^{-1}$.

Let P_1, \dots, P_m be the maximal ideals in R lying over c and $Q_{i,j}$ the maximal ideals in \bar{R} lying over c where the projection of $Q_{i,j}$ on R/I_j is proper. Write $c = \cap_{k=1}^m P_k^{f_k} = \cap_{i,j} Q_{i,j}^{e_{i,j}}$. Then $\text{Aut}_{\bar{R}}(\bar{R} \otimes S) = \times_{j=1}^q \text{Aut}_{R/I_j}(R/I_j \otimes S)$, $\text{Aut}_{R/c}(R/c \otimes S) = \times_{k=1}^m \text{Aut}_{R/P_k^{f_k}}(R/P_k^{f_k} \otimes S)$, and $\text{Aut}_{\bar{R}/c}(\bar{R}/c \otimes S) = \times_{i,j} \text{Aut}_{\bar{R}/Q_{i,j}^{e_{i,j}}}(\bar{R}/Q_{i,j}^{e_{i,j}} \otimes S)$. Let $(\dots, \bar{\sigma}_{i,j}, \dots) \in \text{Aut}_{\bar{R}/c}(\bar{R}/c \otimes S) = \times_{i,j} \text{Aut}_{\bar{R}/Q_{i,j}^{e_{i,j}}}(\bar{R}/Q_{i,j}^{e_{i,j}} \otimes S)$. By hypothesis (a) there is $\sigma_{i,j} \in \text{Aut}_{R/I_i}(R/I_i \otimes S)$ which reduces to $\bar{\sigma}_{i,j}$. For a fixed i , $\{\sigma_{i,j}\}$ determines an element $\sigma_i \in \text{Aut}_{\bar{R}}(\bar{R} \otimes S)$ where we let σ_i be the identity in $\text{Aut}_{R/I_k}(R/I_k \otimes S)$ if k is not any j . Let $\mathcal{U} = \{U_i\}$ be an open cover of $X = \text{Spec}(R)$ where $P_i \in U_j$ if and only if $i = j$. Let $\gamma(\dots, \bar{\sigma}_{i,j}, \dots) = |a| \in H^1(X, \text{Aut}(\mathcal{F}))$ where $a \in Z^1(\mathcal{U}, \text{Aut}(\mathcal{F}))$ is given by $a(i, k) = \sigma_i^{-1} \sigma_k \in \text{Aut}_{\mathcal{O}(U_i \cap U_k)}(\mathcal{O}(U_i \cap U_k) \otimes S)$. Note, $\sigma_i^{-1} \sigma_k$ is defined since $U_i \cap U_k$ contains no P_j so $\mathcal{O}(U_i \cap U_j) \otimes S = \mathcal{O}(U_i \cap U_j) \otimes_{\bar{R}} \bar{R} \otimes S$. It is clear that the definition of γ is independent of the choice of cover and our assignment gives a well-defined map.

Let $\mathcal{U} = \{U_i\}$ be an open cover of X constructed as above and let $\pi : Y \rightarrow X$ be given by restriction. Let $V_i = \pi^{-1}(U_i)$ so $\mathcal{V} = \{V_i\}$ is an open cover of Y . Given $a \in Z^1(\mathcal{U}, \text{Aut}(\mathcal{F}))$, let $\delta(a) \in Z^1(\mathcal{V}, \text{Aut}(\bar{R} \otimes \mathcal{F}))$ by $\delta(a)(i, j) = a(i, j)$. This assignment is well defined since $\mathcal{O}_X(U_i \cap U_j) = \mathcal{O}_Y(V_i \cap V_j)$.

NOTE 2.8. If each R/I_j in Mayer-Vietoris II is integrally closed, $H^1(Y, (\bar{R} \otimes \mathcal{F})) = \{1\}$ by Proposition 2.3. This is the case in all the following examples.

EXAMPLE 2.9. Let \mathbb{Q} denote the rational numbers, let p, q be prime integers, and let ω be a primitive complex p th root of 1. Let $F = \mathbb{Q}(\omega)$ and $R = F[x, y]/(x^p - qy^p(y-1)^p)$. If q is irreducible in $\mathbb{Z}[\omega]$ then by Eisenstein's criterion $x^p - qy^p(y-1)^p$ is irreducible in $F[x, y]$ so R is a one-dimensional Noetherian integral domain. Note that q is irreducible in $\mathbb{Z}[\omega]$ whenever $p \nmid q-1, p \neq q$. Let $S = R[t]/(t^p - q)$. Then S is a finite étale R -algebra which is connected since $t^p - q$ is irreducible in $F[t]$ and $R/(x, y) = F$. Identify x, y with their images in R . The integral closure \bar{R} of R is $R(x/y(y-1))$ and since $(x/y(y-1))^p = q, t^p - q = \prod_{i=0}^{p-1} (t - \omega^i(x/(y(y-1)))) \in \bar{R}[t]$ so $\bar{R} \otimes S \cong \bar{R}^{(p)}$. The maximal ideals lying over c in R are (x, y) and $(x, y-1)$ and the only maximal ideal in \bar{R} lying over (x, y) is (y) , the only maximal ideal in \bar{R} lying over $(x, y-1)$ is $(y-1)$. Since $R/(x, y) \cong F$, and $R/(x, y-1) \cong F$, and $t^p - q$ is irreducible in $F[t]$, R satisfies the hypothesis of Mayer-Vietoris I. But $\text{Aut}_R(S) = C_p$, the cyclic group of order p . $\text{Aut}_{\bar{R}}(\bar{R} \otimes S) = S_p$, the symmetric group on p -letters. $\text{Aut}_{R/c}(R/c \otimes S) = C_p \times C_p$ and $\text{Aut}_{\bar{R}/c}(\bar{R}/c \otimes S) = S_p \times S_p$ so for this example the Mayer-Vietoris I sequence becomes

$$1 \rightarrow C_p \rightarrow S_p \times (C_p \times C_p) \rightarrow S_p \times S_p \rightarrow H^1(X, \text{Aut}(\mathcal{F})) \rightarrow 1. \tag{2.3}$$

Let $K = \{(\tau\rho^{-1}, \tau\sigma^{-1}) \mid \tau \in S_p, \rho, \sigma \in C_p\}$. Then $H^1(X, \text{Aut}(\mathcal{F}))$ is bijective with the coset space $S_p \times S_p / K$ and has order $(p-1)!$. In particular, when $p = 2$ there are no nontrivial twisted forms, when $p = 3$ there is exactly one nontrivial twisted form, and so forth.

EXAMPLE 2.10. (a) Let R be a Noetherian domain with quotient field K , assume R contains a primitive n th root of 1 and I is a fractional R -ideal in K with $I^n = R$. Assume $p(t) = t^n - a \in R[t]$ is a separable polynomial, and $S = R[t]/(p(t))$, $L = K[t]/(p(t))$. Then the subset $T = \bigoplus_{j=0}^{n-1} I^j t^j$ of L is a twisted form of S .

(b) Let R be a reduced Noetherian ring with minimal prime ideals I_1, \dots, I_q and assume the dimension of each R/I_j is one. Let $K = \bigoplus R/I_j$. Assume R contains a primitive n th root of 1 and I is a finitely generated R -submodule of K with $I^n = R$. Assume $p(t) = t^n - a \in R[t]$ is a separable polynomial and $S = R[t]/(p(t))$, $L = K[t]/(p(t))$. Then the subset $T = \bigoplus_{j=0}^{n-1} I^j t^j$ of L is a twisted form of S .

NOTE 2.11. In [Example 2.9](#) if $p = 3$ and $q = 2$ then the nontrivial twisted form T is constructed as in [Example 2.10](#) where the ideal $I = (y, y^{-1}x)$, as one can check by showing the associated cocycle in $H^1(X, \text{Aut}(\mathcal{F}))$ is not a coboundary. Notice T is free as an R -module. Let $\mathcal{F}(S)$ be the set of isomorphism classes of twisted forms of S which are free as R -modules and assume S is free as an R -module. Then there is an exact sequence of pointed sets $1 \rightarrow \mathcal{F}(S) \rightarrow H^1(X, \text{Aut}(\mathcal{F})) \rightarrow H^1(X, \text{Gl}(\mathcal{F}))$. The types of examples given in [Example 2.10](#) all lie in $\mathcal{F}(S)$, but we give in [Example 2.12](#) a twisted form of S whose image in $H^1(X, \text{Gl}(\mathcal{F}))$ is not the identity.

EXAMPLE 2.12. Let $R = \mathbb{R}[x, y]/(y - 1)(y - x^2)$ and $\bar{R} = \mathbb{R}[x, y]/(y - 1) \oplus \mathbb{R}[x, y]/(y - x^2)$, where \mathbb{R} is the set of the real numbers. Then $R = \{(p(x, y), q(x, y)) \in \bar{R} \mid p(1, 1) = q(1, 1); p(-1, 1) = q(-1, 1)\}$. Let $I = \{(p(x, y), q(x, y)) \in \bar{R} \mid p(1, 1) = q(1, 1); p(-1, 1) = -q(-1, 1)\}$. Then I is an R -submodule of \bar{R} , $I^2 = R$, and $R_P \otimes I \cong R_P$ for all prime ideals P of R . Let $p(t) = t^2 + 1$, let $S = R[t]/(t^2 + 1)$, and take $T = R \oplus It$. Then T is a twisted form of S and T is not a free R -module by cancellation [9] so T is a nontrivial twisted form of S . Notice that T is a Galois extension of R with Galois group of order two induced by complex conjugation but since T is not free, T does not have either a normal basis or a primitive element.

EXAMPLE 2.13. Consider T as constructed in [Example 2.12](#) and let $w^2 + 1 \in T[w]$. Then $q(w) = w^2 + 1$ is irreducible in $T[w]$ but for each prime ideal Q of T , $q(w)$ is reducible in $T_Q[w]$. This gives an example of an irreducible separable polynomial over a connected commutative ring which factors into linear factors over the localization at every prime ideal or modulo each maximal ideal. If $r(w) = w^2 - 1$ then $S_1 = R[w]/(q(w))$ is not isomorphic to $S_2 = R[w]/(r(w))$ since $S_2 \cong R \oplus R$ but S_1 and S_2 are locally isomorphic. This is an example of two separable polynomials that are locally isomorphic but not isomorphic (in the sense of [4]).

EXAMPLE 2.14. Let $R = \mathbb{Q}[x, y]/(y - 1)(y - x^2)$ as in [Example 2.12](#). Let $p(t) = t^3 - 3t + 1$ and $S = R[t]/(p(t))$. Using the Mayer-Vietoris sequence for the Picard group, [1] or [5], one can check the torsion part of the Picard group is C_2 . But Mayer-Vietoris II gives $H^1(X, \text{Aut}(\mathcal{F})) = C_3$ so if T is a nontrivial twisted form of S , then T is not isomorphic to $R \oplus It \oplus It^2$ for any fractional ideal of R with $I^3 = R$.

EXAMPLE 2.15. If R is as in [Example 2.14](#) and $p(t) = t^3 - 2$, then one can check that $p(t)$ is separable and $p(t)$ factors into linear factors modulo each minimal prime

ideal of R but $p(t)$ is irreducible in $R[t]$. Hypothesis (a) of Mayer-Vietoris II fails to hold for this example.

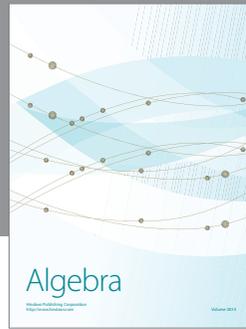
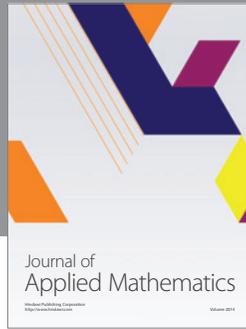
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